OK Geometry

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Extended Abstract

Problem

OK Geometry is a software tool for observing dynamic constructions. During the years of its development, it was a means of exploring the possibilities of automated observation of geometric configurations. We believe the following aspects of automated observation are important:

- How can technology be used to observe geometric properties in the process of proving using dynamic geometry systems and/or automated theorem provers?
- What software functionalities are needed for effective observation of geometric configurations?
- Can students profitably use tools for automated observation in geometry classes?

Motivation

A synthetic proof in geometry consists of a sequence of claims (facts) and arguments arranged in a deductive scheme. In constructing a proof, one often formulates one or more intermediate hypothetical claims and then attempts to provide deductive argument for the claims. The usual way (though not the only way!) of accessing to intermediate claims is by observing the studied geometric configuration. In terms of Toulmin's model of argumentation, observation acts as warrant for the hypothetical intermediate claims. Not being aware of hypothetical intermediate claims can be an obstacle in the process of proving.

Dynamic geometry systems (DGS) create accurate visual representations of geometric constructions, allowing plausible observation of geometric facts. A visual observation can be further backed by magnifying a detail, by measuring, by numerical calculations, by dragging a point or by a combination of these methods. However, all these backings are of little value unless one is aware of hypothetical properties that may be important (not visually observed or otherwise). Automated observation can be helpful in such cases, especially when the intermediate claims involve non-visualised objects or properties that are difficult to see visually (e.g., ratios of geometric quantities).

Some recent DGS incorporate automated theorem provers of various kinds. Time will tell if and how they can become part of school mathematics. Nonetheless, they are a useful tool for students to test the correctness of hypothetical claims. Automated observation can be a rich source of hypotheses to prove, new discoveries, proof exercises, etc.

State of the art

OK Geometry is a tool for observing properties of dynamic constructions. It generates plausible hypotheses about the properties of geometric configurations and, in this sense, can be also considered an aid for proving them. To the author's knowledge, OK Geometry is the only DGS specifically designed for the observation of dynamic constructions (by no means were ad hoc computer programs created for the discovery of geometric properties based on numerical checks).

All DGS contain several functionalities that allow visual and/or numerical checks of properties of a dynamic construction. Some of them also offer possibilities to provide mathematically sound proofs.

Automated proving of geometric facts is based on a variety of methods, not all of them are suitable or successful FOR all cases or types of tasks. We first mention the group of algebraic methods, included, for example, in GeoGebra, GCLC, JGEX. A proof obtained by algebraic methods is undoubtedly a valid verification, but from the perspective of school mathematics one may question the didactic value of such a proof. According to Hanna, verification is only one, not necessarily the most relevant, function of proof; often what matters is understanding why something is true and developing hypothetico-deductive argumentation. But perhaps we should say that using and interpreting algebraic provers requires different competencies than working out classical proofs. Provers from the second group base the arguments on some geometric quantities, e.g., full-angle, area, Pythagorean difference. Such provers are included, for example, in CGLC and JGEX. Another type, called randomised prover, is included in Cinderella. For a given configuration of points and lines, it generates a list of properties of the configuration. Mathematically valid verifications of the properties are obtained by considering a finite number of randomised examples of the configuration.

Perhaps the most interesting prover from the perspective of school mathematics is the GDD method in the JGEX software. For a given dynamic construction, the method generates a list of properties of the configuration along with synthetic proofs that are reasonably in line with standard geometric reasoning in schools. The method considers the set of points of a construction and the initial set of properties, which are essentially the construction steps. Then, using a fixed set of lemmas, new properties for the initial set of points are derived from the assumption and added to the set of properties. The process is repeated on the sequentially augmented set of properties (for the initial set of points) until no new properties are added. As a result, we get a list of properties of the construction – and each property is accompanied by a synthetic and illustrated proof. JGEX thus finds and proves facts of a configuration – but only those that are derivable from the used lemmas on the initial set of points. Even in fairly simple cases, some relevant properties (in particular, those related to proportions) may be missing from the list. For facts that are not in the list JGEX attempts to generate a GDD proof by adding an additional point to the construction.

We also mention here the MrGeo software, designed for documenting proofs. It does not look for properties of dynamic constructions, nor does it create proofs, but it allows various modes of presenting proofs: as a flowchart, as two-column proof, and in paragraph mode. OK Geometry uses some techniques (presentation modes) of MrGeo. One way to use OK Geometry for proving a property of a construction is to select appropriate properties from the observed properties and organise them in a deductive argument. The proof can then be presented in flowchart, paragraph or two-column mode.

OKG can observe dynamic constructions created with several DGS (e.g. GeoGebra, Cabri, Cinderella) and complements them in this sense. But the really interesting thing is the interaction of OK

Geometry with DGS with automated proving capabilities. The properties, observed with OK Geometry, can serve as hypotheses that can be verified with one of the proving tools.

Contribution

The basic operating principle of OK Geometry is quite simple and can be described as constant dragging of all free points of a dynamic construction. Instead of a single instance of a dynamic construction, OKG constantly generates several instances obtained by randomly positioning all free points of the construction. On these random instances, OK Geometry numerically checks a variety of geometric properties. A property of the configuration is considered to be observed if it passes the corresponding numerical test for all the instances of the construction. (More precisely, only the basic properties are numerically checked, other properties are derived using text-processing methods.) The result of the observation is a list of observed properties.

Automated observation is thus based on numerical checks and on the ability to generate any number of instances of a construction. The simple basis of the observation method has several benefits.

- A wide range of geometric properties can be systematically observed (not just those in the focus of the user).
- The observation can take into account not only the geometric objects in the construction, but as well additional objects (several thousand) derived from the objects of a construction.
- The method can be applied also to the analysis of geometric quantities. OK Geometry, for example, observes whether up to 6 geometric quantities in a construction satisfy a polynomial relation of up to 4-th degree. In other words, we can 'observe algebraic expressions' of geometric quantities in the considered construction.
- When considering specific types of constructions, specific objects and properties can be taken into account. For example, when observing objects in relation to a reference triangle, hundreds and thousands of centres, lines, circles, conics, triangles, and cubics of the reference triangle are considered.

Observation results with OK Geometry are not due to sophisticated methods for detecting properties, but to effective application of many different tests to a large number of objects. As a consequence, some technical and operational issues have to be considered.

From the technical point of view, it is important to control the probability of observational errors. Such errors are due to working with finite precision arithmetic, to approximation errors (when data are obtained by optimisation procedures or by solving linear systems of considerable size), as well as to the large amount of tests. For example, a configuration of 15 points gives rise to 930930 pairs of angles, and can very easily happen that the ratios of two of these pairs happen to be the equal (up to the computational precision) by chance.

To make the observation process useful and effective from the user's point of view, the following was considered:

1. The observation is (usually) performed on a dynamic construction. Since users are often used to creating constructions on a preferred DGS, OK Geometry allows to importing dynamic constructions from various DGS (e.g. GeoGebra, Cinderella, Cabri).

2. Whenever we want to observe configurations that we do not know how to construct, we can resort to implicit constructions. Put simply, an implicit construction of a configuration consists of a 'partial construction' and one or more requirements on the instances of the construction. OK Geometry creates and observes instances of the partial construction that satisfy the additional requirements.
3. Observation fairly simple configuration may result in a huge number of observed properties. Therefore, mechanisms for organising and searching properties are required. Care must be taken to taken to get rid of trivial properties in the list.
4. When observing and analysing objects with respect to a reference triangle, indeed many triangle specific objects are considered: centres, lines, circles, conics, cubics, triangle transformations. For example, OK Geometry may observe that a point under consideration lays on the Euler line of the Brocard triangle of the reference triangle. To deal with the multitude of triangle objects, OK Geometry provides commands for constructing triangle objects, methods for relating triangle objects to a considered object, and an illustrated glossary of triangle objects.

We conclude with some pedagogical remarks about two examples of applying automated observation in school setting.

The first example consists of a group of experiments on the effects of using automated observation in lower secondary schools when learning to prove facts in geometry. Students were given a (simple) geometric situation and a claim. They were asked to: 1. list potentially true (relevant) properties of the configuration; 2. provide a proof of the given claim by selecting appropriate properties from the list; 3. argue the chosen properties and organise the properties into a proof. Students solved some tasks using paper and pencil and other tasks using OK Geometry with prepared dynamic constructions.

The experiments showed that when using OK Geometry students identified many relevant properties that they were not aware of when working with paper and pencil. Not being aware of a relevant property can be an obstacle when creating a proof. However, contrary to our expectations, the results of automated observation were not a significant help for students, as they had difficulty dealing with the (extensive) list of properties and selecting which properties to choose and how to use them in the proof.

As a result of these experiments we have developed a simple method to be used in the initial learning of proving. Students are presented with a simple configuration, a claim to be proved, and a set of cards with pictorial representation of the properties to be used in the proof. The students must provide arguments for the properties and organise them into a proof. The method is essentially a paper and pencil method, OK Geometry or another DGS is used only to create the pictorial representations of the properties used.

A second example is an experiment in which we explored whether prospective mathematics teachers (two-subject teachers in lower secondary schools) can profitably use DGS, tools for automated observation and proving tools. The research was integrated into a 15 hour geometry seminar. Students were first given a brief introduction to topics: 1. Algebraic proofs and proving in GeoGebra, 2. Observing dynamic constructions with OK Geometry, 3. Proving methods of JGEX. Each student was then given a theme for the seminar in triangle geometry with some open-ended tasks to solve. The assignment incorporated the following elements:

1. Create a dynamic representation of a theorem or relevant concept. Present a synthetic proof of a theorem. Students demonstrated understanding of the proof by transforming the proof from paragraph mode to the two-column or flowchart mode.
2. Use OK Geometry to observe relevant properties related to the considered concept or theorem.
3. Report the results of using the automated theorem provers for some of the facts observed with OK Geometry.

The results of the experiment were satisfactory in terms of students' ability to use the software, to understand the role and meaning of automated observation, to understand the role and meaning of the computer generated proofs, and also in terms of students' attitude towards the use technology in geometry tasks.

Main Idea

The educational community recognised the didactic potential of DGS almost immediately after its dissemination. The strongest and most utilised aspect of DGS is the dynamic presentation of concepts. The deductive aspect of geometry was given somewhat less attention. Dragging, measuring, numerical checks of properties can promote hypothetical-deductive reasoning, but only if students are aware of the properties to be considered. OK Geometry is a tool that observes and brings to attention various properties of dynamic constructions, but does not prove them. We have presented some important features that we have found to be important in the observation process. Automated observation also interacts well with DGS with theorem provers, because it provides a rich set of hypothetical properties that require for solid proofs.