Applications of quantifier elimination in geometry

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Introduction

Let φ be a first-order formula. By quantifier elimination (QE) we mean:

Find a formula θ without quantifiers such that

$$\varphi \Leftrightarrow \theta$$

In particular, for a quantifier-free ψ find θ without quantifiers such that

$$\exists \psi \Leftrightarrow \theta$$

Usually θ is easily computable, while φ might be uncomprehensive and often assumes search over an infinite domain.

Many famous theorems have the form of quantifier elimination.

Examples

- $\exists x(x^2 + px + q = 0) \Leftrightarrow p^2 - 4q \geqslant 0, \quad p, q \in R.$
- Let $a, b, c \in R$. Then

  There is a triangle with sides $a, b, c \Leftrightarrow$
  $a + b > c$, $a + c > b$, $b + c > a$ and $abc > 0$.
- Kronecker - Capelli theorem: If $AX = B$ is a system of linear equations over R, then: $\exists X\ AX = B \Leftrightarrow \operatorname{rang}(AB) = \operatorname{rang}B$.
- Euler criterion: Let p be an odd prime. Then in the field $\mathrm{GF}(p)$, $a \in \{1, 2, \ldots, p-1\}$, $\exists x\ x^2 = a \Leftrightarrow a^{\frac{p-1}{2}} = 1$.
- Let $r = r(t)$ be the parametric equation of a space curve C. Then C is planar if and only if torsion of C is equal to 0, i.e.

  There is a plane containing $C \Leftrightarrow \frac{(r' \times r'') \cdot r'''}{\|r' \times r''\|^2} = 0$

Precise definition

Our previous definition of QE is informal.

Formal definition of QE for first order logic assumes first order language L and first order theory T formulated in L.

Definition Theory T admits QE if and only if for each formula φ of L there ia a formula θ of L without quantifiers such that

$$T \vdash \varphi \Leftrightarrow \theta$$

Usually it is assumed, but not necessarily, that L is a recursive set and T has recursive axiomatization.

There are two main approaches for proving that a theory T admits QE, and studying such theories and their models.

The first one is model-theoretic, the second one is the explicit construction of QE algorithm for T.

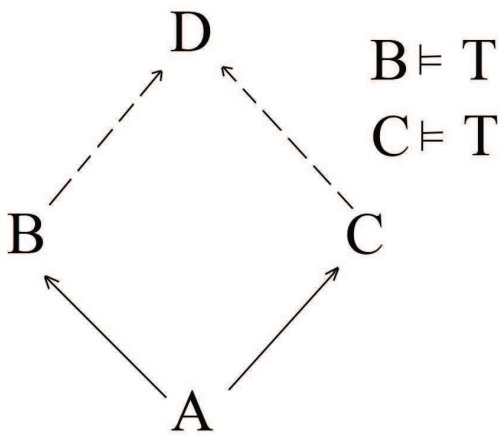
Model-theoretic approach

The first methods assumes techniques of model theory, a branch of logic.

It is very fruitful, but on the other hand very non-constructive.

For example, in model theory one proves that the following two statements for a first order theory T are equivalent.

1. T admits elimination of QE.
2. Every diagram of the following sort can be completed as shown



In model theory strong axioms AC and GCH of set theory are used. However, it is proved:

If QE for T is proved in model theory, then there is an effective algorithm of QE in T.

G. Sacks, *Saturated model theory*, 1972.

C.C. Chang, J. Keisler *Model theory*, 1973.

History

QE for a theory T implies it's decidability and in many cases decidability is proved by use of QE, not mentioning QE explicitly.

Probably the oldest QE procedure is **Fourier - Motzkin elimination**, an algorithm for eliminating variables from a system of linear inequalities.

The algorithm is named after J. Fourier who proposed the method in 1826 and T. Motzkin who re-discovered it in 1936.

Later it was generalized to ordered Abelian groups (Gurevich, 1964), and further to various commutative rings and modules.

Presburger arithmetic is the first-order theory of the natural numbers with addition and $<$. M. Presburger proved in 1929 that this theory is complete and decidable by use of QE.

There are automatic theorem provers for Presburger arithmetic. Such example is the *Coq* proof assistant system, while the *Isabelle* proof assistant contains a verified quantifier elimination procedure.

History

QE for real closed fields (RC) (field of real numbers) and closed fields (field of complex numbers) was first demonstrated by Alfred Tarski in his seminal paper *A Decision Method for Elementary Algebra and Geometry* (1951).

However, Tarski's algorithm was highly inefficient in practice. Abraham Seidenberg (1954) gave a simpler procedure and since then QE for the theory of real closed fields is usually attributed as "Tarski-Seidenberg".

Perhaps the most efficient general algorithm currently known, and the first actually to be implemented on a computer, is the **Cylindrical Algebraic Decomposition** (CAD) method introduced by George Collins (1975).
Recent contributions in this field which speed up QE procedures for RC in computer implementations belong to W. Weispfenning, Wen-Tsu Wu (triangulation technic) and others, often using Groebner bases.

QE for RC

By transforming geometrical problems into analytic form, already Tarski proved decidability of elementary geometry. Since then many algorithms for solving problems in geometry were offered and computer programs for automatic deduction implemented.

Some notable implementation of QE for RC have Wolfram Mathematica (WM) programming system and Maple, but there are more specialized programs.

We present here some uses of QE for RC in solving tangible problems on algebraic varieties and in geometry of various natures.

For easier understanding we formulate theorems and examples for real algebraic curves and surfaces. All theorems and examples are easily generalized for algebraic varieties of arbitrary dimensions. We also assume that varieties appearing in next examples are rational.

Our programming tool is WM. We remind that QE commands of WM are **Resolve** and **Reduce**.

Classical problems in computer aided geometric design

Given a parametric rational polynomial hypersurface $\mathcal{H}$ (In real applications, curves in R^2 and surfaces in R^3),

- Find the implicit polynomial equation of $\mathcal{H}$ (implicitization);
- Find the parameter values corresponding to the coordinates of a point known to lie on $\mathcal{H}$ (inversion). If, for example, parametric equations of a surface $\mathcal{H}$ in R^3 are: $x = F(u, v), \quad y = G(u, v), \quad z = H(u, v)$, then

Implicitization: QE applied on $\exists u \exists v (x = F(u, v) \wedge y = G(u, v) \wedge z = H(u, v))$ should produce the implicit polynomial equation of $\mathcal{H}$.

Inversion: QE applied on $\exists x \exists y (x = F(u, v) \wedge y = G(u, v) \wedge z = H(u, v))$ produces relation between parameters u and v if the point (x, y) belongs to $\mathcal{H}$.

Enneper surface (minimal surface theory), Enneper 1864

The parametric equations for the Enneper surface $\mathcal{E}$ and implicitization are represented by the following code in WM.

```
F= u (1 - u^2/3 + v^2)/3
G= -v (1 - v^2/3 + u^2)/3
H= (u^2 - v^2)/3
Reduce [Exists [{u, v}, x==F && y==G && z==H], Reals]
```

On an average PC this code produces an output in 0.3 seconds:

$$x^2 == Root[-27y^6 - 9y^4z - 162y^4z^2 - 48y^2z^3 + 135y^4z^3 - 240y^2z^4 - 64z^5 + 432y^2z^5 - 144y^2z^6 + 128z^7 - 64z^9 + (81y^4 + 18y^2z - 48z^3 + 702y^2z^3 + 240z^4 + 432z^5 + 144z^6)\#1 + (-81y^2 - 9z + 162z^2 + 135z^3)\#1^2 + 27\#1^3 \;\&, 1]$$

This readily gives the implicit polynomial equation of $\mathcal{E}$:

$$-27y^6 - 9y^4z - 162y^4z^2 - 48y^2z^3 + 135y^4z^3 - 240y^2z^4 - 64z^5 + 432y^2z^5 - 144y^2z^6 + 128z^7 - 64z^9 + (81y^4 + 18y^2z - 48z^3 + 702y^2z^3 + 240z^4 + 432z^5 + 144z^6)x^2 + (-81y^2 - 9z + 162z^2 + 135z^3)x^4 + 27x^6 = 0$$

Otherwise, implicitization of Enneper surface is not so easy task for a human.

Boundness problem

If $\mathcal{H}$ is a real algebraic variety, for example a surface given by a polynomial $F(x, y, z)$, the boundness of $\mathcal{H}$ can be tested by

$$\exists m \forall xyz(F(x, y, z) = 0 \Rightarrow |x| + |y| + |z| \leqslant m).$$

Therefore, by Tarski-Seidenberg algorithm we immediately have:

Theorem

If $\mathcal{H}$ is an algebraic variety over the field of real algebraic numbers then the boundness of $\mathcal{H}$ is effectively tested.

If in the above formula two variables on the right side are omitted then the boundness along third axis can be tested. If $\exists m$ is omitted, the effective bound R can be obtained. Here are some examples.

Boundness problem

$\mathcal{H}_1$: $F[x\_, y\_] := 2x^4 - 8x^2y^2 + 7y^4 - 3x - y^3 - 20$

$\mathbf{Resolve[ForAll[\{x, y\}, Implies[F[x, y] == 0, Abs[x] + Abs[y] \leq m]], Reals]}$

This code yields **False**, so $\mathcal{H}_1$ is unbounded. If $\mathcal{H}_1$ in above code is replaced by

$\mathcal{H}_2$: $F[x\_, y\_] := 2x^4 - 7x^2y^2 + 7y^4 - 3x - y^3 - 20$, the code outputs:

$$m \geq \text{Root}[-15479680 - 12397500\#1 - 4855671\#1^2 - 815046\#1^3 + 21527485\#1^4 + 10036848\#1^5 + 1289248\#1^6 - 1148252\#1^7 - 7728780\#1^8 - 673040\#1^9 - 7560\#1^{10} - 26656\#1^{11} + 10976\#1^{12}\&, 2].$$

Hence, $\mathcal{H}_2$ is bounded and any $m \geq R$ can serve as a bound, where R is the second root of the polynomial $f(x)$,

$$f(x) = -15479680 - 12397500x - 4855671x^2 - 815046x^3 + 21527485x^4 + 10036848x^5 + 1289248x^6 - 1148252x^7 - 7728780x^8 - 673040x^9 - 7560x^{10} - 26656x^{11} + 10976x^{12}$$

The constant $R \approx 6.5311$ is the best bound and the computation took about 1.4 sec.

Flatness of 3D algebraic curves

Suppose an algebraic curve C in R^3 is given parametrically by

$$x = X(t), \quad y = Y(t), \quad z = Z(t), \qquad t \in R.$$

We can test if C is flat by

$$\exists abcd \, \forall t \, (aX(t) + bY(t) + cZ(t) + d = 0 \wedge (a^2 + b^2 + c^2 + d^2 \neq 0))$$

A similar formula can be constructed if C is given by intersection of two algebraic surfaces.
If we check

$$\forall t \, (aX(t) + bY(t) + cZ(t) + d = 0))$$

as output we shall obtain either $a = 0, b = 0, c = 0, d = 0$ (case non-flat), or a set of algebraic equations involving a, b, c, d which solution set would give us a 3D plain containing C.

Hence, **flatness problem for algebraic curvesis is decidable and numerically soluble**.

Embedding problem

From differential geometry we know that C is flat if it's torsion is equal to 0. There is a formula for curve torsion, so we can test the flatness of C in this way, too.

However, in a similar way as above we can test if any algebraic variety in R^k is embedded into an algebraic variety of certain kind in R^m. For example we can test if C is a spherical curve.

Hence the **general embedding problem for algebraic varieties is decidable** and in principle numerically soluble.

On Internet we found advices that one can test the flatness problem C by checking if 3, or 4 randomly chosen points on C are coplanar.

Of course this is not true, but one can prove for general embedding problem that there is a finite set of points depending only on degrees of involved polynomials and dimensions of the spaces R^m and R^n.

Equidistant curves

Distance $d(A, \mathcal{V})$ between a point $A \in R^n$ and a set $\mathcal{V} \subseteq R^n$ is defined as $\inf_{x \in \mathcal{V}} d(x, A)$.

If $d > 0$, loci K of points in R^2 having the same distance d to $\mathcal{V} \subseteq R^2$ is called the **equidistant curve** of $\mathcal{V}$.

If $\mathcal{V}$ is a 2-dimensional rational algebraic variety given by a finite set of algebraic equations and inequalities $J(u, v)$ then the equidistant curve K at distance $d > 0$ is the result of QE applied to:

$$\exists uv(J(u, v) \wedge D2(u, v, x, y) = d), \quad \text{where } D2(u, v, x, y) = (u - x)^2 + (v - y)^2.$$

The next WM code finds equidistant curve $\mathcal{K}$ to an ellipse at the distance $d = 1$.

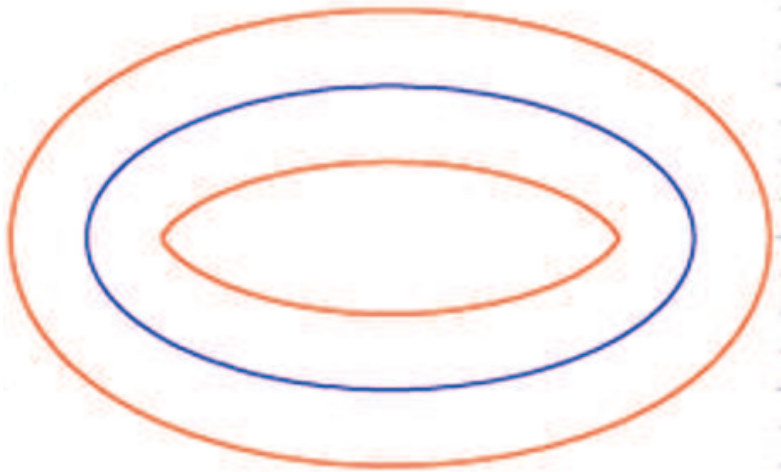
The variety $\mathcal{K}$ decomposes into two curves, one inside the ellipse, the other one outside, see the enclosed picture.

Equidistant curves to ellipse

```
D2[u_, v_, x_, y_] := (u - x)^2 + (v - y)^2; (* squared distance between two points *)
J[x_, y_] := x^2/16 + y^2/4 - 1 == 0; (*ellipse*)
Resolve[Exists [{u, v}, J[u, v] && D2[u, v, x, y] == 1], Reals]
```

The curve $\mathcal{K}$ is determined by the equation $H(x, y) = 0$, where

$$H(x,y) = 18225 - 17010y^2 - 1431y^4 + 200y^6 + 16y^8 + \left(-6804 + 4590y^2 - 570y^4 + 40y^6\right)x^2 + \left(918 - 390y^2 + 33y^4\right)x^4 + \left(-52 + 10y^2\right)x^6 + x^8.$$



Equidistant curves to ellipse

IMO Problem 1

On the International mathematical Olympiad in 1964 the following problem is posed.

If a, b, c are lengths of the sides of a triange, prove:

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}S(a, b, c)$$

where

$$S(a, b, c) = \frac{1}{4}\sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}$$

Solution od IMO Problem 1

WM Solution

(* Olympiad III.2 *)

Clear[a, b, c];

$$S(a\_, b\_, c\_) := \frac{1}{4}\sqrt{(a+b+c)\,(-a+b+c)\,(a-b+c)\,(a+b-c)}$$

$$\text{Resolve}\left[\forall_{\{a,b,c\}}\left(a+b>c \wedge b+c>a \wedge a+c>b \wedge 0<a\,b\,c \Rightarrow a^2+b^2+c^2 \geq 4\sqrt{3}\ S(a,b,c)\right), \mathbb{R}\right]$$

True

$$\text{Resolve}\left[\forall_{\{a,b,c\}}\left(a+b>c \wedge b+c>a \wedge a+c>b \wedge 0<a\,b\,c \Rightarrow a^2+b^2+c^2 \geq d\,S(a,b,c)\right), \mathbb{R}\right]$$

$d \leq 4\sqrt{3}$

$$\text{Resolve}\left[\exists_{\{a,b,c\}}\left(a+b>c \wedge b+c>a \wedge a+c>b \wedge 0<a\,b\,c \Rightarrow a^2+b^2+c^2 = d\,S(a,b,c)\right), \mathbb{R}\right]$$

True

$$\text{Resolve}\left[\exists_{\{c\}}\left(a+b>c \wedge b+c>a \wedge a+c>b \wedge a>0 \wedge b>0 \wedge c>0 \wedge a^2+b^2+c^2 = 4\sqrt{3}\ S(a,b,c)\right), \mathbb{R}\right]$$

$$b>0 \wedge a = \text{Root}\left[\#1^4 - 2\,\#1^2\,b^2 + b^4\ \&,\ 3\right]$$

$$\text{Solve}\left[b^4 - 2\,b^2\,x^2 + x^4 = 0,\ x\right]$$

$$\{\{x \to -b\},\ \{x \to -b\},\ \{x \to b\},\ \{x \to b\}\}$$

IMO Problem 2

On the International mathematical Olympiad in 1967 the following problem is posed.

If a, b, c are lengths of the sides of a triange, prove:

$$a^2(-a+b+c)+b^2(a-b+c)+c^2(a+b-c)\leqslant 3abc$$

WM solution

```
(* Olympiad  VI.2*)

(Clear[a, b, c];
Resolve[∀{a,b,c} (a + b > c ∧ b + c > a ∧ a + c > b ∧ a > 0 ∧ b > 0 ∧ c > 0 ⇒
                  a^2 (-a + b + c) + b^2 (a - b + c) + c^2 (a + b - c) ≤ 3 a b c), ℝ])
True
Resolve[∀{a,b,c} (a + b > c ∧ b + c > a ∧ a + c > b ∧ a > 0 ∧ b > 0 ∧ c > 0 ⇒
                  a^2 (-a + b + c) + b^2 (a - b + c) + c^2 (a + b - c) ≤ a b c d), ℝ]
d ≥ 3
```

Appolonian circle

The equidistant curve for two varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ is the loci of points K heaving the same distances to $\mathcal{V}_1$ and $\mathcal{V}_2$.

For example, equidistant curve for two circles (in plane) are the second order curves.

If three disjoint circles are given, the intersection of equidistant curves to these circles gives the center of the Appolonian circle, that one touching all three circles.

QE can be used to prove that Appolonian circle exists and to construct it as well.

Appolonian circle

Problem setting

```
(* Apollonian circles:
   There is a circle K taching three given circles K1, K2, K3 *)
(*                                                            *)
(* K1: x^2 + y^2 = 1                                          *)
(* K2: (x - c)^2 + y^2 = r^2                                  *)
(* K3: (x - a)^2 + (y - b)^2 = R^2                            *)
(* K:  (x - p)^2 + (y - q)^2 = ρ^2                            *)
(* Conditions for circles K1, K2, K3 not to
  intersect:
  I1: 1 +r < c,
  I2: (1 + R)^2 < a^2 + b^2,
  I3: (r + R)^2 < (a-c)^2 - b^2
      Sum of radiuses < centers distance *)
(* Conditions for circle K to touch K1, K2, K3:
  J1: (1 + ρ)^2 == p^2 + q^2,
  J2: (r + ρ)^2 == (p - c)^2 + q^2,
  J3: (R + ρ)^2 == (p - a)^2 + (q - b)^2
      Sum of radiuses == centers distance *)
```

Appolonian circle

WM program code

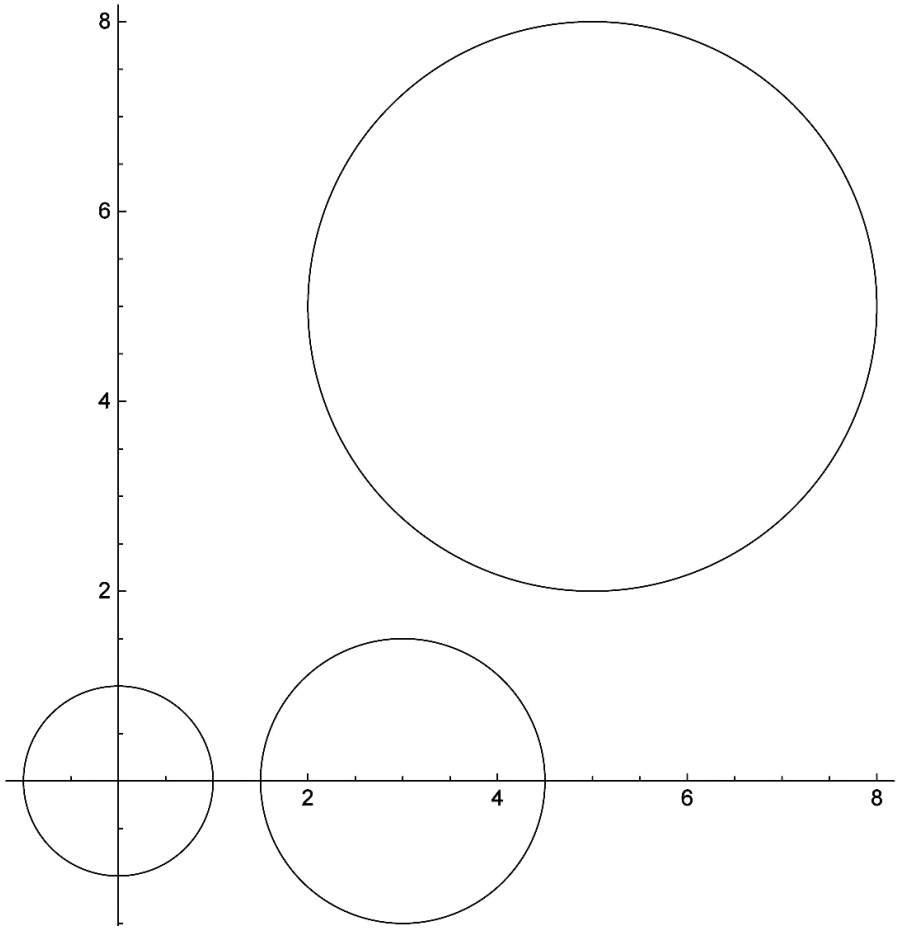
```
Clear[x, y, a, b, c, r, R, I1, I2, I3, C1, C2]

I1 (r_, c_) := -c + r + 1;
I2(R_, a_, b_) := -a^2 - b^2 + (R + 1)^2;
I3(r_, R_, a_, b_, c_) := - (a - c)^2 - b^2 + (r + R)^2;
C1(x_, y_, c_, r_) := (c^2 - 2 c x - (r - 1)^2)^2 - 4 (r - 1)^2 (x^2 + y^2);
C2 (x_, y_, a_, b_, c_, r_, R_) := - (r - 1) (a^2 + b^2) + 2 x (a (r - 1) - c (R - 1)) +
                2 b (r - 1) y + c^2 (R - 1) - (r - 1) (R - 1) (r - R);
Resolve[∀{a,b,c,r,R} (b ≠ 0 ∧ r ≠ 1 ∧ R ≠ 1 ∧
                I1(r, c) < 0 ∧ I2(R, a, b) < 0 ∧ I3(r, R, a, b, c) < 0 ∧ 0 < r ∧ 0 < R ⇒
                  ∃{x,y} (C1(x, y, c, r) == 0 ∧ C2(x, y, a, b, c, r, R) == 0)), Reals]
```

Appolonian circle example

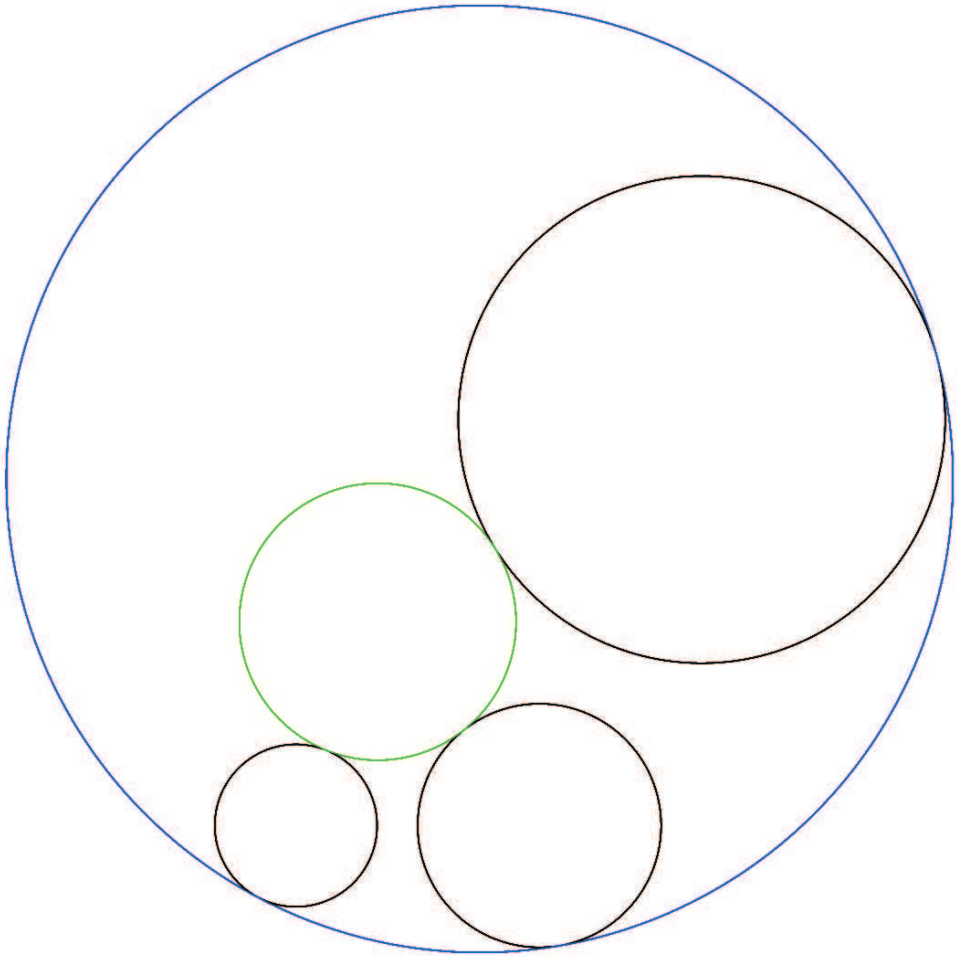
Three circles

```
Graphics[{Circle[{0, 0}, 1], Circle[{c, 0}, r], Circle[{a, b}, R]}, Axes → True]
```



Appolonian circle example

Solution



c := 3
r := 1
a := 5
b := 2
R := 4

Open problem

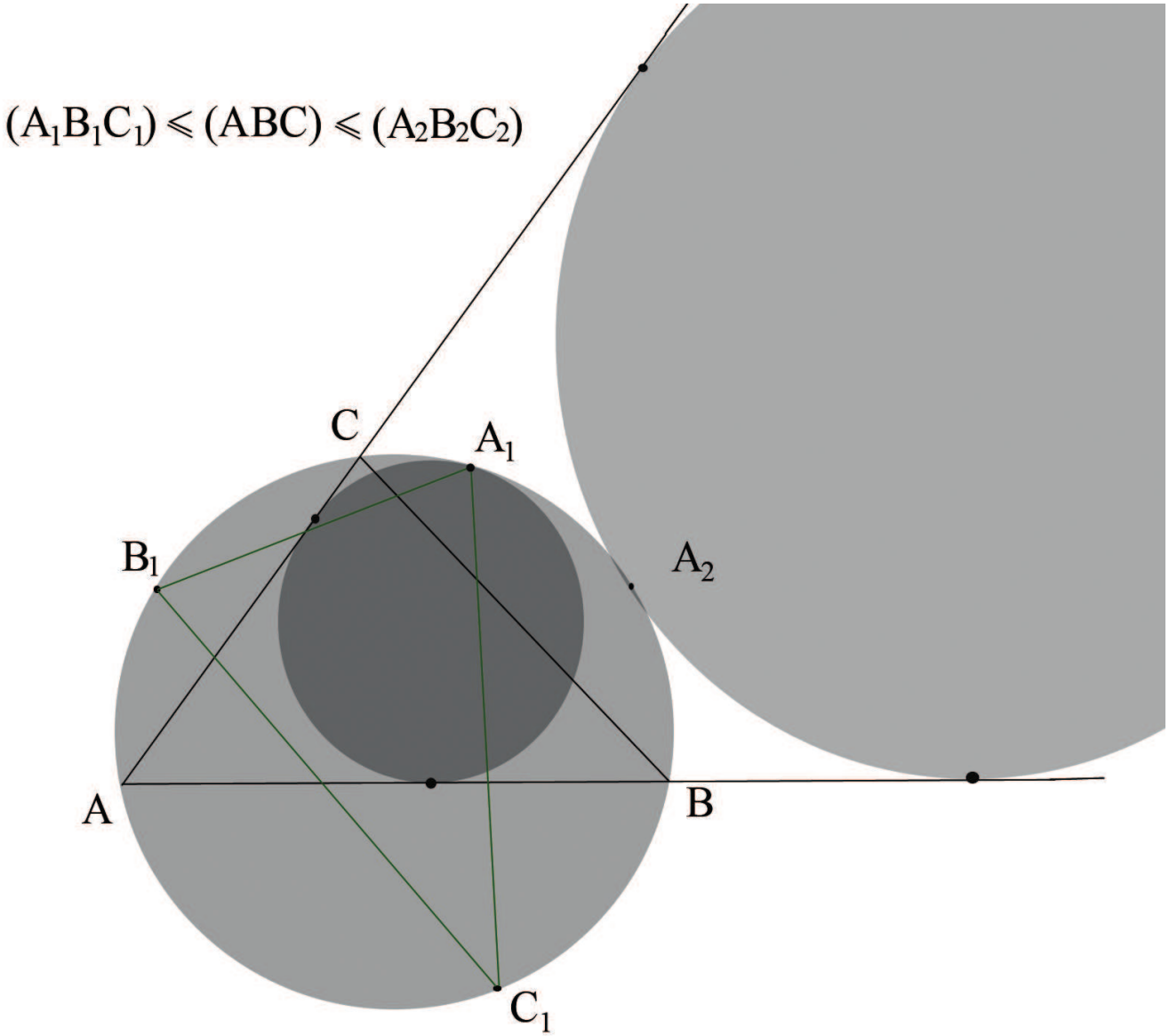
- **Proposed by Le Thai Hoang.**
  (1) Given a triangle ABC. A circle touches sides AB and AC and touches internally the circumcircle of ABC at A_1. A circle touches the sides AB and AC and touches externally the circumcircle of ABC at A_2. Similarly, define B_1, B_2, C_1, C_2. Let (X) denote the area of X. Prove that

  (A_1 B_1 C_1) <= (ABC) <= (A_2 B_2 C_2).

  (2) Given a tetrahedron ABCD. A sphere touches the three faces containing A and touches internally the circumsphere of ABCD at A_1. A circle touches the three faces containing A and touches externally the circumsphere of ABCD at A_2. Similarly, define B_1, B_2, C_1, C_2, D_1, D_2. Let (X) denote the volume of X. Prove that

  (A_1 B_1 C_1 D_1) <= (ABCD) <= (A_2 B_2 C_2 D_2).

Open problem



Open problem

Comment by Murat Aygen and Naoki Sato.
I am happy to say that we have made some progress on problem (1). Aygen writes: Let w be the circle tangent to AB and AC, and tangent internally to the circumcircle. Then a similitude centred at A send the incircle to w. Another similitude centred at A_1 sends w to the circumcircle.
According to a well-known theorem, the product of two similitudes is also a similitude, and its centre is collinear with the centre of the two similitudes. Therefore, AA_1 passes through a centre of similitude of the incircle and the circumcircle (we can tell which one by keeping track of the signs of the similitudes). By symmetry, BB_1 and CC_1 also pass through this centre, so they all concur. Similarly, AA_1, BB_2, and CC_2 concur, at the other centre of similitude of the incircle and circumcircle.
One can also show that AA_1, BB_1, and CC_1 concur using inversion, and furthermore, that they concur at the isogonal conjugate of the Nagel point of triangle ABC, centre 56 in the Encyclopedia of Triangle Centres. **The problem then "reduces" to showing that**

$$u^5 v + uv^5 + u^5 w + uw^5 + v^5 w + vw^5 + 6(u^3 v^2 w + u^3 v w^2 + u^2 v^3 w + u^2 v w^3 + u v^3 w^2 + u v^2 w^3) \geq 4(u^4 vw + uv^4 w + uvw^4) + 2(u^3 v^3 + u^3 w^3 + v^3 w^3) + 24u^2 v^2 w^2$$

for all non-negative reals u, v, and w. **I have not been able to do this - yet.**

Open problem Solution

```
Resolve[
 ForAll[{u, v, w}, (u ≥ 0 ∧ v ≥ 0 ∧ w ≥ 0) ⇒ u^5 v + u v^5 + u^5 w + u w^5 + v^5 w + v w^5 +
      6 (u^3 v^2 w + u^3 v w^2 + u^2 v^3 w + u^2 v w^3 + u v^3 w^2 + u v^2 w^3)
    ≥ 4 (u^4 v w + u v^4 w + u v w^4) + 2 (u^3 v^3 + u^3 w^3 + v^3 w^3) + 24 u^2 v^2 w^2]]
```

Out[ ]= True

```
Clear [u, v, w]
AbsoluteTiming[
  Resolve[
    ∀{u,v,w} (u ≥ 0 ∧ v ≥ 0 ∧ w ≥ 0 ⇒
          u^5 v + u^5 w +
          6 (u^3 v^2 w + u^3 v w^2 + u^2 v^3 w + u^2 v w^3 + u v^3 w^2 + u v^2 w^3) + u v^5 + u w^5 + v^5 w + v w^5 ≥
          4 (u^4 v w + u v^4 w + u v w^4) + 2 (u^3 v^3 + u^3 w^3 + v^3 w^3) + 24 u^2 v^2 w^2)
  ]]
```

Out[ ]= {7.47438, True}

Appolonian problem for three ellipses

QE can be used to solve the generalized problem, to find the Appolonian circle touching three disjoint ellipses, or more generally, three disjoint rational algebraic varieties.

For three ellipses P, Q and S, the idea is first to find their rational parametrization in parameter t in order to reduce the number of quantified variables in QE.

Further, QE is applied on $\forall t(\mathrm{DP2}(x, y, t) \geqslant d)$, where $\mathrm{DP2}(x, y, t)$ is the distance function of a point (x, y) from P.

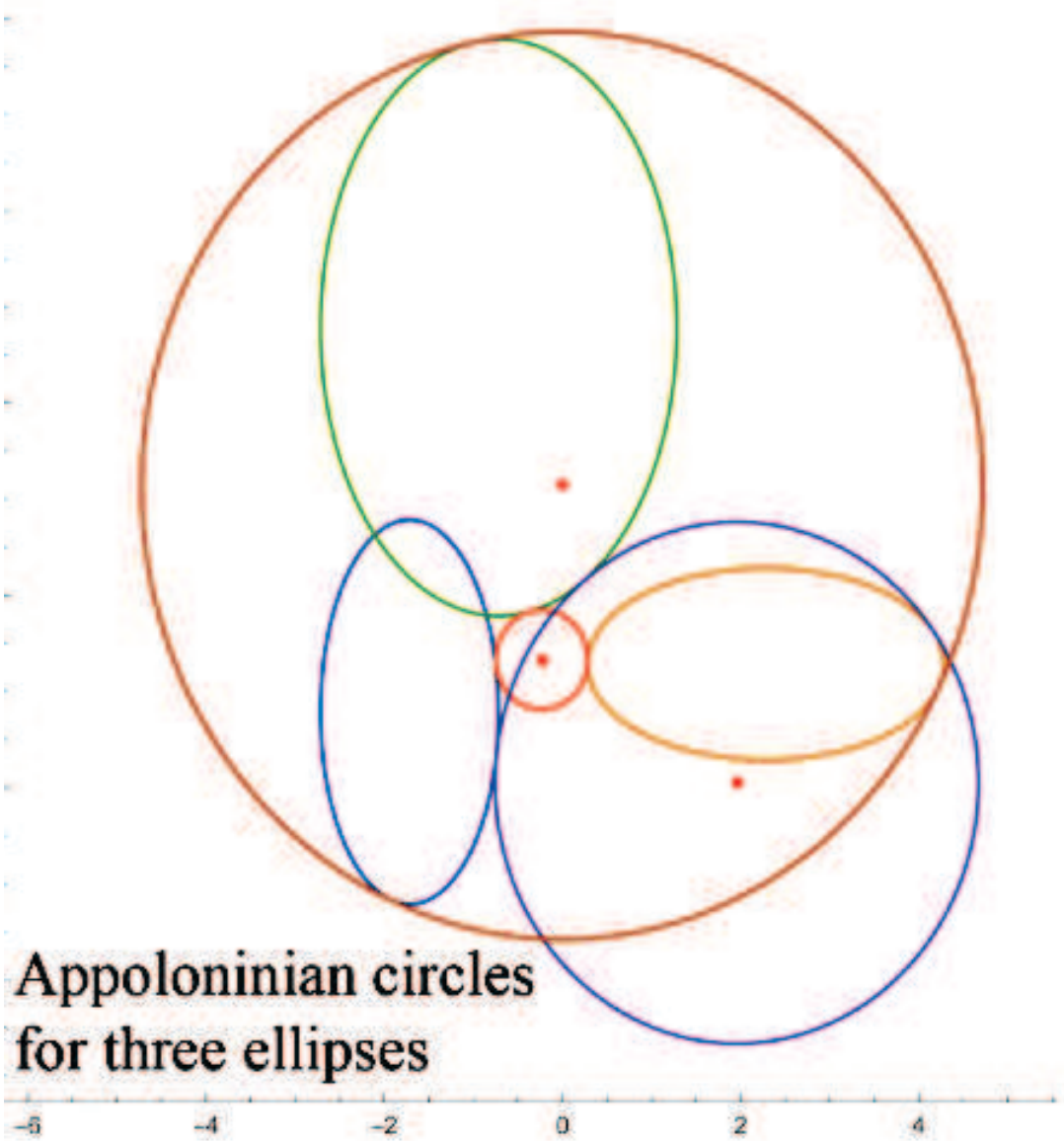
A polynomial $p(x, y, d)$ of order 8 is obtained.

Similarly we obtain polynomials $q(x, y, d)$ and $s(x, y, d)$ resp. for ellipses Q and S. If a point (x, y) has the same distances from P and Q, the polynomials p and q have the common variable d.

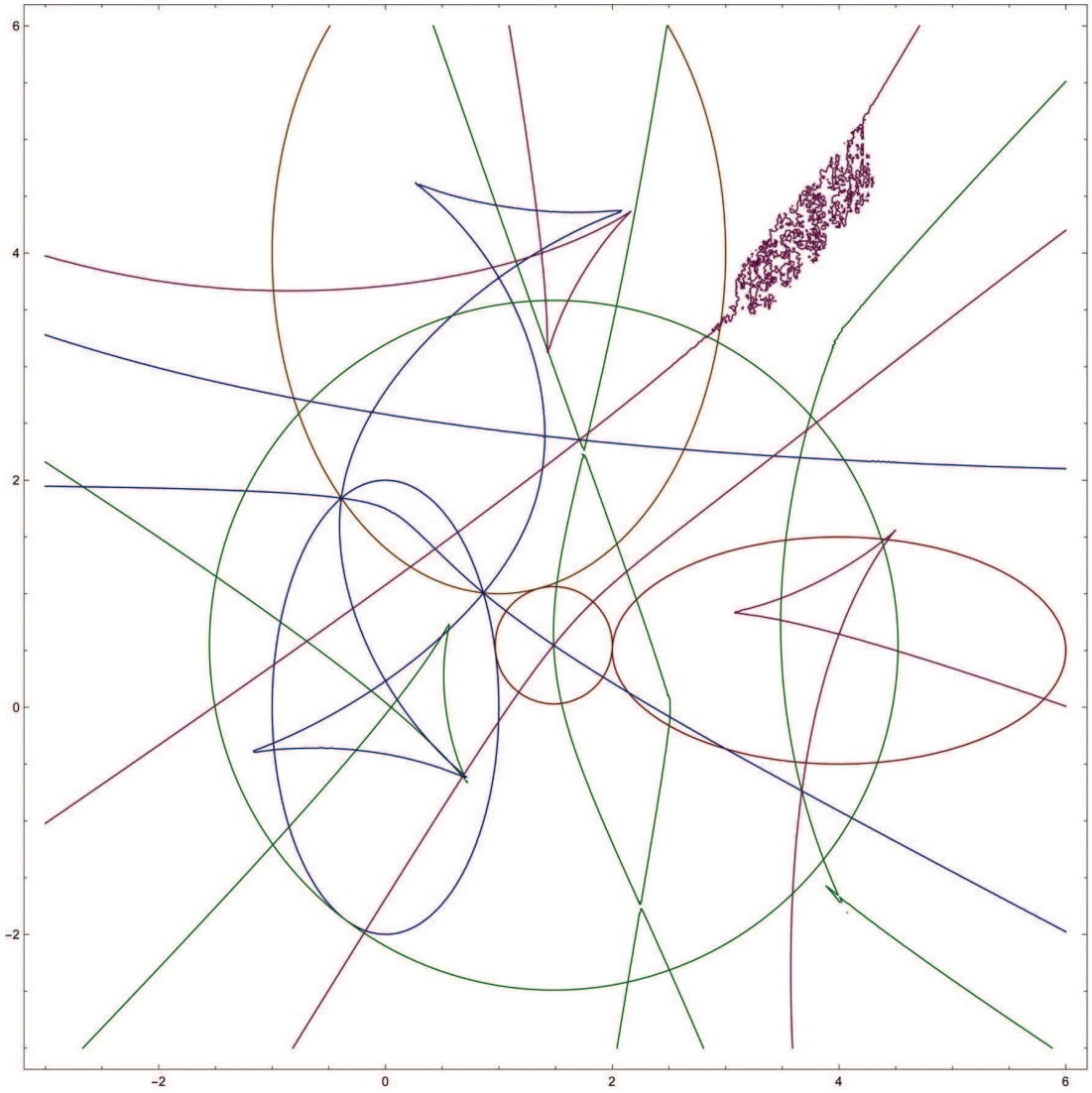
Eliminating d we got a polynomial $e_{PQ}(x, y)$ which represents equidistant curve from P and Q.

Polynomial $e_{PQ}(x, y)$ has degree 812.

Appolonian circle for three ellipses



Anomaly



Reducing number of variables

QE has hyperexponential complexity (2^{2^n}) so it is very ineffective for problems described with large number of variables.
Some procedures for reducing the number of variables:

1. Find canonical set $\mathcal{K}$ of structures for a problem P:
   If statement φ is proved for structures in $\mathcal{K}$ then φ holds for all structures involved in P.
   Example: Most geometric problems are invariant in respect to isometry and homothety. Hence for triangle problems we can take $\mathcal{K}$ (in analytic approach) triangles in first quadrant with vertices $B = (0, 0)$, $C = (1, 0)$ and $A = (x, y)$, $x > 0, y > 0$.
   QE is now applied only on 2 variables (x, y) instead of 6.
2. Apply QE on parameterized equation instead of implicite form.

Example: Circle is given by $x^2 + y^2 = 1$, but also by
$x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$, $t \in R$.

Implicit programming

The goal of this presentation was to find solutions of certain problems using methods of mathematical logic and some specialized software **SpecS** such as Wolfram's Mathematica.

The idea is to think of first order logic as of a programming language and use it to obtain a description D of a problem P and leave D to **SpecS** to solve it. The code D is called the **implicit program** for the problem P.

There are two types of solutions:

1. Elimination of quantifiers in formulas from D (Wolfram's Mathematica).
2. Building a (finitary) model for the set of formulas in D.

While the first method is essentially based on the application of QE for algebraically closed and real closed fields, the second one usually uses the skolemization and building of Herbrand universe for formulas depicted in D (Mace4).

Telescope

They ordered telescope, but...

"Hmm... Lemme check that purchase order again."