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Automated proof of Ramsey theorem via symbolic computation

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Outline

- 1. What is Ramsey's Theorem
- 2. Hand proofs of Ramsey Theorem
- 3. How to prove Ramsey's Theorem by symbolic computation?
- 4. Prove R(3,4)=9 via symbolic computation
- 5. Discussion

1. What is Ramsey's Theorem

- Definition: the Ramsey number *R(s, t)* is the number of vertices in the smallest complete graph which, when 2-colored red and blue, must contain a red *K_s* or a blue *K_t* , where we denote the complete graph on n vertices by *K_n* .

Ramsey Theorem. *For any two natural numbers, s and t, there exists a natural number, $R(s,t) = n$, such that any 2-colored complete graph of order at least n, colored red and blue, must contain a monochromatic red K_s or blue K_t.*

1. Ramsey Theorem R(3,3)=6

- For any complete graph with 6 vertices, use red and blue to color its edges in arbitrary way, then, there must be a red K_3 or a blue K_3. The 6 is the smallest number with this property.

1. General Ramsey's Theorem:

- For any positive integers $s,t > 1$, $R(s,t) < +\infty$.

- $R(4,4) = 18$,
- $43 \le R(5,5) \le 48$, $102 \le R(6,6) \le 165$.

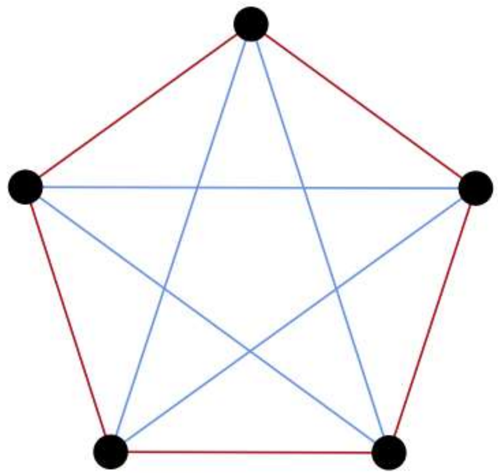
Table 1 The 9 known non-trivial Ramsey numbers

r	3	3	3	3	3	3	3	4	4
s	3	4	5	6	7	8	9	4	5
$R(r,s)$	6	9	14	18	23	28	36	18	25

2. A hand proof of Ramsey Theorem

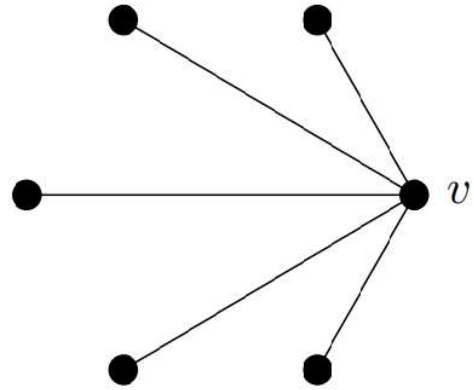
- For the original Ramsey Theorem *R(3,3)=6* :

Proof. First, we show that $R(3,3) > 5$ (or $R(3,3) \geq 6$) by exhibiting a complete graph on 5 vertices that does not contain a red K_3 or blue K_3:

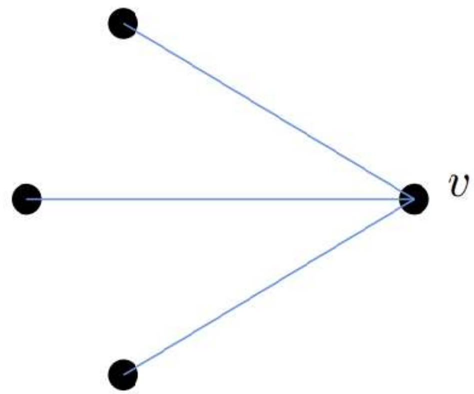


We now show that K_6 must always contain a red K_3 or blue K_3. Recall that this is equivalent to the statement of the Friends and Enemies Puzzle.

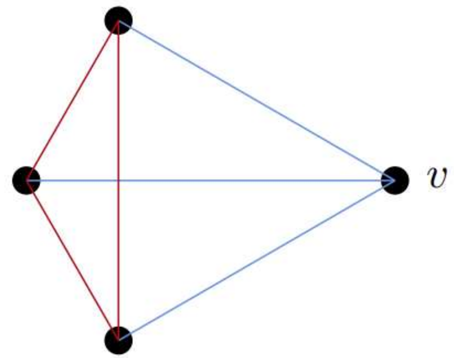
First, pick any vertex v and consider the edges incident to it:



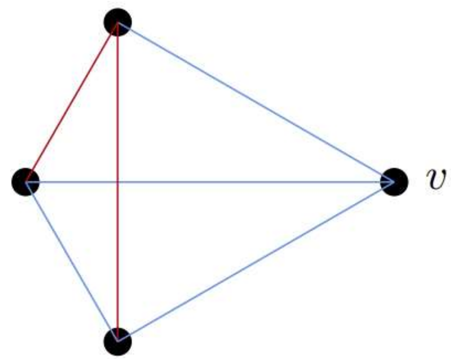
Since there are 5 edges and only 2 possible colors for each edge, by the pigeonhole principle, at least 3 of these edges must have the same color. Without loss of generality, assume there are 3 blue edges connecting v to 3 other vertices.



Consider the K_3 subgraph generated by the 3 adjacent vertices. If all edges in the subgraph are red, then we have found a red K_3.



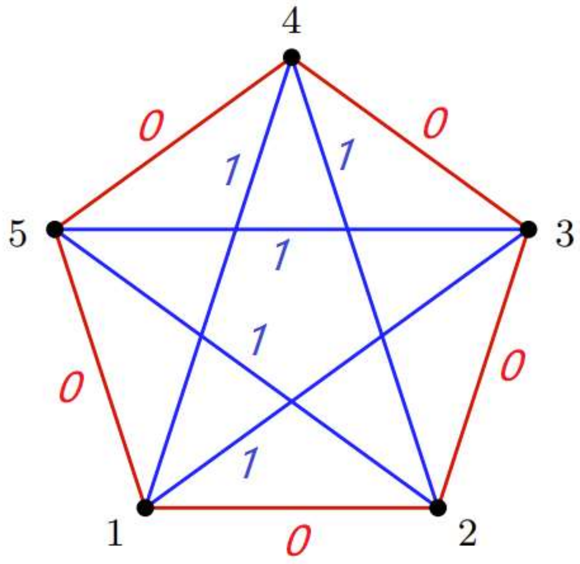
Otherwise, at least one of the edges must be blue. This edge completes a blue K_3 with the original set of 3 blue edges incident to v.



Therefore, $R(3, 3) = 6$. $\square$

3. How to prove Ramsey’s Theorem by symbolic computation?

- Step 1: Use 0,1 to represent the edge color red and blue. If edge between the vertex i and the vertex j is red, then $x_{ij}=0$, if it is blue, then $x_{ij}=1$.

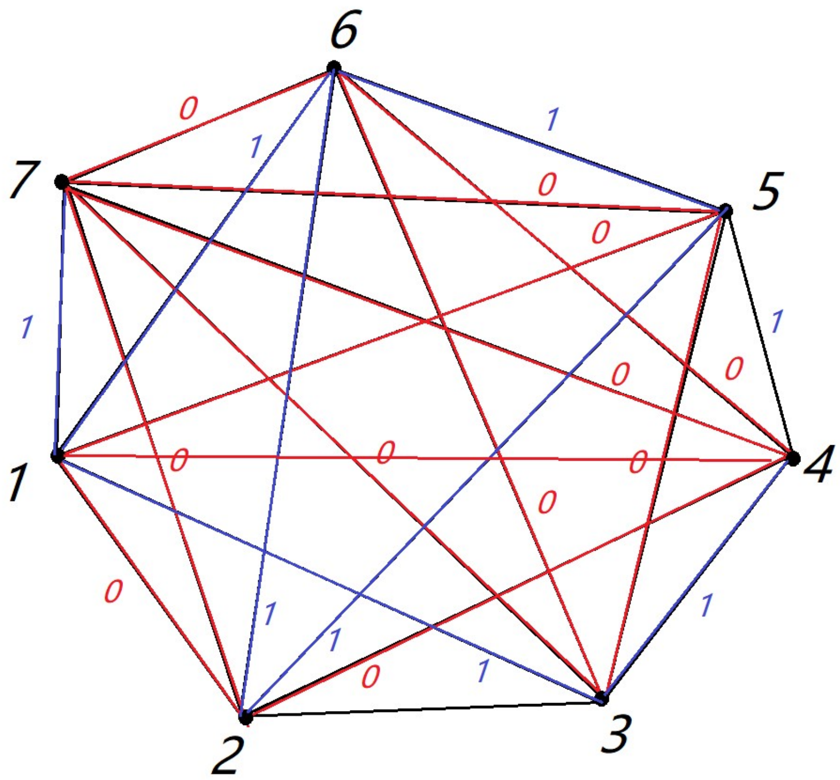


x12=0, x13=1, x14=1, x15=0,

x23=0, x24=1, x25=1,

x34=0, x35=1,

x45=0



In this K_7,

9 edges are blue, and 12 edges are red.

x13, x16, x17, x23, x25, x26,
x34, x45, x56 = 1

x12, x14, x15, x24, x27, x35, x36,
x37, x46, x47, x57, x67 = 0

3. How to prove Ramsey's Theorem by symbolic computation?

- Step 2: Assume that a complete graph *K_n* is colored by 0 (red) and 1 (blue). *x_{ij}* takes 0 if the edge between i, j is red, and *x_{ij}* takes 1 if the edge between *i, j* is blue. Define two functions *I* and *J* as follows:

$$J = \prod_{1 \le i < j < k \le \mathrm{n}} f_3(x_{ij}, x_{jk}, x_{ik}), \; I = \prod_{1 \le i < j < k \le \mathrm{n}} g_3(x_{ij}, x_{jk}, x_{ik}).$$

where

$$f_r(z_1, z_2, \ldots, z_{r(r-1)/2}) := z_1 + z_2 + \cdots + z_{r(r-1)/2},$$
$$g_s(z_1, z_2, \ldots, z_{s(s-1)/2}) := 1 - z_1 z_2 \cdots z_{s(s-1)/2},$$

3. How to prove Ramsey's Theorem by symbolic computation?

- Step 3: Compute the polynomial *J x I*, using the simplify rules:

$$\{x_{12}^2 = x_{12}, x_{12}^3 = x_{12}, x_{12}^4 = x_{12}, \ldots, x_{45}^2 = x_{45}, x_{45}^3 = x_{45}, x_{45}^4 = x_{45}\},$$

- If the result is *J x I = 0* (zero polynomial), we claim R(3,3)<=n;

and if the result is not a zero polynomial, we claim R(3,3)>n.

3. How to prove Ramsey's Theorem by symbolic computation?

- For example, for the complete graph K_5, we have:

$$I = 1 - x_{12}x_{13}x_{14} - x_{12}x_{13}x_{15} - \cdots - x_{34}x_{35}x_{45} \\ + \cdots + 4x_{12}x_{13}\cdots x_{15}x_{23}\cdots x_{45},$$

$$J = 3(x_{12}x_{13}x_{23}\cdot x_{45} + x_{12}x_{14}x_{24}\cdot x_{35} + \cdots + x_{34}x_{35}x_{45}\cdot x_{12}) \\ + \cdots + 60x_{12}x_{13}\cdots x_{15}x_{23}\cdots x_{45}.$$

- The multiplication J x I:

$$J \times I = 32x_{12}x_{13}x_{24}x_{35}x_{45} + 32x_{12}x_{14}x_{23}x_{35}x_{45} + \cdots - 384x_{12}x_{13}\cdots x_{15}x_{23}\cdots x_{45}.$$

This polynomial contains 218 monomials, and

$$J \times I\,(0,1,1,0,0,1,1,0,1,0) = 32,$$

hence, R(3,3) > 5.

3. How to prove Ramsey's Theorem by symbolic computation?

- For the complete graph K_6, we have

- J is a polynomial with 5,789 monomials,

- I is a polynomial with 5,395 monomials, (see next page)

- And J x I = zero polynomial after simplification.

$$J(x_{12}, x_{13}, \ldots, x_{56}) = \prod_{1 \le i < j < k \le 6} f_3(x_{ij}, x_{ik}, x_{jk})$$

$$= 9 \sum_{1 \le i < j < k \le 6} \prod_{\substack{1 \le l_1 < l_2 \le 6 \\ l_1, l_2 \ne i,j,k}} x_{l_1 l_2} \cdot x_{ij} x_{ik} x_{jk} + \cdots + 26250768 \prod_{1 \le i < j \le 6} x_{ij},$$

$$I(x_{12}, x_{13}, \ldots, x_{56}) = \prod_{1 \le i < j < k \le 6} g_3(x_{ij}, x_{ik}, x_{jk})$$

$$= 1 - \sum_{1 \le i < j < k \le 6} x_{ij} x_{ik} x_{jk} - \cdots - 3 \prod_{1 \le i < j \le 6} x_{ij},$$

here the reduced form of J has $5,789$ monomials, the highest degree of which is 15, and the lowest degree is 6, and the reduced form of I has $5,395$ monomials, the highest degree of which is also 15.

to expand the product $H = J \times I$,

using $x_{ij}^2 = x_{ij}$ $(1 \le i < j \le 10)$ to simplify the result, we obtain $H \equiv 0$ finally, which implies that $R(3,3) \le 6$.

4. Prove R(3,4)=9 via symbolic computation

- Theoretically, the same method can be used if there is no intermediate computation explosion.

- For the complete graph K_9, the number of variables x_{ij} is 9x8/2=36. Direction computation of J,I polynomials is too complicated.

- Method:

4. Prove R(3,4)=9 via symbolic computation

In the first step, we write the chromatic variable V in the following form:

$$
\begin{aligned}
&x_{12},\\
&x_{13}, x_{23},\\
&x_{14}, x_{24}, x_{34},\\
&\cdots\ \cdots\ \cdots\ \cdots\\
&x_{18}, x_{28}, x_{38}, \ldots, x_{78},\\
&x_{19}, x_{29}, x_{39}, \ldots, x_{79}, x_{89}
\end{aligned}
$$

and rearrange the vertices $1, 2, \ldots, 8$ so that

$$x_{19} = \cdots = x_{k9} = 0,\ x_{k+1,9} = \cdots, x_{89} = 1,\ \ (0 \le k \le 8)$$

then divide the original problem for computing $H = J \times I$ into the following 9 sub-problems $(P_k)\,(k = 0, 1, \ldots, 8)$:

4. Prove R(3,4)=9 via symbolic computation

Sub-Problem (P_k) : $H = \texttt{mult}(J, I)$ where

J and I are defined in (23), and

$x_{1j} = 0\,(1 \leq j \leq k),\;\; x_{j9} = 1\,(k+1 \leq j \leq 9).$

In each task (P_k), we compute the multiplication of some factors of J, I and search certain complete subgraph $K = \{i_1, i_2, \ldots, i_p\}$, formed by p vertices $1 \leq i_1 < i_2 < \cdots < i_p \leq 9$ of K that satisfies $H_K = J|_K \times I_K = 0$, here, J_K, I_K are defined as follows:

$$J_K := \prod_{\substack{1 \leq i<j<k \leq 9 \\ i,j,k \in K}} F_{i,j,k}, \quad I_K := \prod_{\substack{1 \leq i<j<k<l \leq 9 \\ i,j,k,l \in K}} G_{i,j,k,l}.$$

here

$$F_{i,j,k} := f_3(x_{ij}, x_{jk}, x_{ik}) = x_{ij} + x_{jk} + x_{ik},$$

$$G_{i,j,k,l} := g_4(x_{ij}, x_{ik}, x_{il}, x_{jk}, x_{jl}, x_{kl}) = 1 - x_{ij}\, x_{ik}\, x_{il}\, x_{jk}\, x_{jl}\, x_{kl}.$$

Clearly, if K is any complete subgraph of K_9, then H_K is a divisor of H, and therefore, $H_K = 0$ implies that $K = 0$. Thus, the key to solve each sub-problem is to find subgraph K with relatively small number of vertices so that $H_K = 0$.

Theorem 6. (1) *When* $k = 0, 1, 2$, *the subgraph* $K = \{3, 4, 5, 6, 7, 8, 9\}$ *satisfies* $H_{K_7} \equiv 0$;

(2) *When* $k = 4, 5, 6, 7, 8$, *the subgraph* $K = \{1, 2, 3, 4, 9\}$ *satisfies* $H_{K_5} \equiv 0$;

(3) *When* $k = 3$, *the following statement is true:*

$$\texttt{not}\left(H_{K_4(1,2,3,9)} \equiv 0\right) \wedge \texttt{not}\left(H_{K_6(4,5,6,7,8,9)} \equiv 0\right) \Longrightarrow H_{K_8(1,2,3,4,5,6,7,8)} \equiv 0.$$

Key idea of the proof:

The three cases mentioned in Theorem 6 are found by computational experiment.

5. Discussion

(1) An improvement of Ramsey's Theorem:

Theorem 5. $R_2(3,3) = 6$, *i.e., for any 2-coloring chromatic scheme*

$$V = (x_{12}, x_{13}, \ldots, x_{16}, x_{23}, \ldots, x_{26}, x_{34}, \ldots, \ldots, x_{56})$$

of the complete graph K_6, *there exist* $1 \le i < j < k \le 6, 1 \le i' < j' < k' \le 6$ *so that both* (i, j, k) *and* (i', j', k') *are colored by single color.*

5. Discussion

- (2) for three colors, we may use 0, -1, 1 to represent the colors, and the following characteristic polynomials:

$$f_0(z_1, z_2, \ldots, z_k) = z_1^2 + z_2^2 + \ldots + z_k^2,$$
$$f_1(z_1, z_2, \ldots, z_k) = z_1 + z_2 + \ldots + z_k - k,$$
$$f_{-1}(z_1, z_2, \ldots, z_k) = z_1 + z_2 + \ldots + z_k + k,$$

- to represent that a complete subgraph of Kn is colored by one special color.

5. Discussion

- Can quantum computers be use to do computation for simplify the polynomials generated in the proof of Ramsey’s theorem?

- (This is the end page)

- Thank you very much!

- The authors are very sorry that they are not able to join the conference in Belgrade Serbia.

- Zhenbing Zeng at Shanghai, 2023-09-18