# The Companion and Bézout Subresultants of Bernstein Polynomials

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# Outline

# Outline

Notations

- $M_{s\times t}$ where $s \le t$ and $M_i$ is the $i$-th column of $M$
- $\operatorname{detp} M := \sum_{i=0}^{t-s} \det \begin{bmatrix} M_1 & \cdots & M_{s-1} & M_{t-i} \end{bmatrix} \cdot x^i$
- $F(x) = f_m x^m + \cdots + f_0$ where $f_m \neq 0$
- $G(x) = g_n x^n + \cdots + g_0$ where $g_n \neq 0$
- $M_{F,k}^{(p)} = \begin{bmatrix} f_m & f_{m-1} & \cdots & f_0 & & \\ & \ddots & \ddots & & \ddots & \\ & & f_m & f_{m-1} & \cdots & f_0 \end{bmatrix}_{(n-k)\times(m+n-k)}$
- $M_{G,k}^{(p)} = \begin{bmatrix} g_n & g_{n-1} & \cdots & g_0 & & \\ & \ddots & \ddots & & \ddots & \\ & & g_n & g_{n-1} & \cdots & g_0 \end{bmatrix}_{(m-k)\times(m+n-k)}$

# Review on Subresultants

**Definition**

The $k$-th subresultant polynomial of $F$ and $G$ is defined as

$$S_k(F,G) := \operatorname{detp} \begin{bmatrix} M_{F,k}^{(p)} \\ M_{G,k}^{(p)} \end{bmatrix}$$

**Proposition (Equivalent determinant form, Li 06')**

We have $S_k(F,G) = \det \begin{bmatrix} M_{F,k}^{(p)} \\ M_{G,k}^{(p)} \\ X_k^{(p)} \end{bmatrix}$, where $X_k^{(p)} = \begin{bmatrix} -1 & x & & \\ & \ddots & \ddots & \\ & & -1 & x \end{bmatrix}$

## Motivation

- Extend **standard basis** to **Bernstein basis**
- Develop **subresultant** formulas for **Bernstein polynomials**

# Problem Statement

## Definition

Let $\boldsymbol{w}_s = \left[w_{s,0}(x), \quad w_{s,1}(x), \quad \ldots, \quad w_{s,s}(x)\right]^T$ where

$$w_{s,i}(x) = \binom{s}{i}(1-x)^{s-i}x^i \text{ for } 0 \le i \le s$$

Then $\boldsymbol{w}_s$ is called the **Bernstein basis** of $\mathbb{Q}_s[x]$. A **Bernstein polynomial** of degree $s$ is a linear combination of $w_{s,i}$'s, i.e., $\sum_{i=0}^{s} c_i w_{s,i}(x)$.

## Problem

**Input:** $\deg(F) = m$, $\deg(G) = n$, $m \ge n$, and $0 < k < n$

$$F = \sum_i a_i w_{m,i}(x) = a_0\binom{m}{0}(1-x)^m x^0 + \cdots + a_m\binom{m}{m}(1-x)^0 x^m$$

$$G = \sum_i b_i w_{n,i}(x) = b_0\binom{n}{0}(1-x)^n x^0 + \cdots + b_n\binom{n}{n}(1-x)^0 x^n$$

**Output:** $S_k(F,G)$ in $\boldsymbol{w}_k = \left[w_{k,0}(x), \quad w_{k,1}(x), \quad \ldots, \quad w_{k,k}(x)\right]^T$

# Outline

# Related Works on Subresultants for Bernstein Polynomials

- Winkler et al. 00', Sylvester resultant matrix
- Winkler et al. 00', **Companion resultant matrix**
- Bini et al. 04', **Beźout resultant matrix**
- Wu & Chen 15', Connection between Sylvester and Bézout resultant matrix
- Winkler et al. 16', Sylvester subresultant matrix
- Tan & Yang 23', Sylvester subresultant polynomials

# Scaled Bernstein Basis

## Definition

Let $\overline{\boldsymbol{w}}_s(x) = \left[\overline{w}_{s,0}(x), \quad \overline{w}_{s,1}(x), \quad \ldots, \quad \overline{w}_{s,s}(x)\right]^T$ where

$$\overline{w}_{s,i}(x) = (1-x)^{s-i}x^i \text{ for } 0 \le i \le s.$$

Then $\overline{\boldsymbol{w}}_s$ is called the **scaled Bernstein basis** of $\mathbb{Q}_s[x]$. Moreover, a **scaled Bernstein polynomial** of degree $s$ is a linear combination of $\overline{w}_{s,i}$'s.

# Companion Resultant Matrix

Given $P = \sum_{i=0}^{n} p_i \overline{w}_{n,i}(x) \in \mathbb{Q}[x]$ where $p_n \neq 0$, the **companion matrix** of $P$ in scaled Bernstein basis is defined as

$$C_P := E^{-1} A$$

where

$$E = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ -\frac{p_0}{p_n} & -\frac{p_1}{p_n} & \cdots & -\frac{p_{n-2}}{p_n} & -\frac{p_{n-1}}{p_n} + 1 \end{bmatrix}, \quad A = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ -\frac{p_0}{p_n} & -\frac{p_1}{p_n} & -\frac{p_2}{p_n} & \cdots & -\frac{p_{n-1}}{p_n} \end{bmatrix}$$

## Definition

The **companion resultant matrix** of $F$ and $G$ in scaled Bernstein basis is defined as

$$N(F, G) := G(C_F)$$

# Bézout resultant matrix

## Definition

The **Bézout resultant matrix** of the polynomials $F$ and $G$ in Bernstein basis is defined as an $m \times m$ matrix $B^{(b)}(F,G)$ such that

$$\frac{\det \begin{bmatrix} F(x) & G(x) \\ F(y) & G(y) \end{bmatrix}}{x-y} = \boldsymbol{w}_{m-1}^T(x) \cdot B^{(b)}(F,G) \cdot \boldsymbol{w}_{m-1}(y).$$

The entries of $B^{(b)}(F,G)$ are:

- $B_{i,1}^{(b)} = \frac{m}{i}(a_i b_0 - a_0 b_i)$ for $1 \leq i \leq m$
- $B_{i,j+1}^{(b)} = \frac{m^2}{i(m-j)}(a_i b_j - a_j b_i) + \frac{j(m-i)}{i(m-j)} B_{i+1,j}$ for $1 \leq i, j \leq m-1$
- $B_{m,j+1}^{(b)} = \frac{m}{(m-j)}(a_m b_j - a_j b_m)$ for $1 \leq j \leq m-1$

# Outline

# Companion Subresultants of Bernstein Polynomials

## Theorem (Main Result)

*For $k < n$, we have*

$$S_k(F,G) = c \cdot \det \begin{bmatrix} P_k N(F,G) \\ U_k X_{m-1}^{(b)} \end{bmatrix}$$

*where*

- $c = \left( \sum_{i=0}^{m} a_i \binom{m}{i} (-1)^{m-i} \right)^{n-k}$
- $X_{m-1}^{(b)} = \begin{bmatrix} x & -(1-x) & & \\ & \ddots & \ddots & \\ & & x & -(1-x) \end{bmatrix}_{(m-1)\times m}$
- $P_k = \begin{bmatrix} \binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{k} & & \\ & \ddots & \ddots & & \ddots & \\ & & \binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{k} \end{bmatrix}_{(m-k)\times m}$
- $U_k = \begin{bmatrix} \binom{m-k-1}{0} & \binom{m-k-1}{1} & \cdots & \binom{m-k-1}{m-k-1} & & \\ & \ddots & \ddots & & \ddots & \\ & & \binom{m-k-1}{0} & \binom{m-k-1}{1} & \cdots & \binom{m-k-1}{m-k-1} \end{bmatrix}_{k\times(m-1)}$

# Companion Subresultant of Bernstein Polynomials

- $P_k = \begin{bmatrix} \binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{k} & & & \\ & \ddots & \ddots & & \ddots & & \\ & & \binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{k} \end{bmatrix}_{(m-k)\times m}$
- $U_k = \begin{bmatrix} \binom{m-k-1}{0} & \binom{m-k-1}{1} & \cdots & \binom{m-k-1}{m-k-1} & & & \\ & \ddots & \ddots & & \ddots & & \\ & & \binom{m-k-1}{0} & \binom{m-k-1}{1} & \cdots & \binom{m-k-1}{m-k-1} \end{bmatrix}_{k\times(m-1)}$

**Remark:**

- $P_k$ is the matrix such that $\overline{\boldsymbol{w}}_k = P_k \overline{\boldsymbol{w}}_{m+n-k-1}$;
- $U_k$ is the matrix such that $\overline{\boldsymbol{w}}_{k-1} = U_k \overline{\boldsymbol{w}}_{m-2}$.

# An Illustrative Example

Consider $k = 2$ and

$$F(x) = 2\, w_{4,0}(x) - \frac{1}{2}\, w_{4,1}(x) - \frac{1}{2}\, w_{4,2}(x) - \frac{5}{4}\, w_{4,3}(x) - w_{4,4}(x)$$
$$G(x) = 4\, w_{3,0}(x) - 2\, w_{3,1}(x) - 2\, w_{3,2}(x) - \; w_{3,3}(x),$$

1. By calculation, $c = 1$.
2. Construct $C_F$.

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 2 & 3 & 6 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & 3 & 5 \end{bmatrix}$$

$$C_F = E^{-1}A = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & -\frac{4}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{4}{5} & \frac{4}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{4}{5} & -\frac{1}{5} & \frac{6}{5} \end{bmatrix}$$

# An Illustrative Example (Cont.d)

3. Compute $N(F,G)$.

$$N(F,G) = G(C_F) = \begin{bmatrix} -4 & 6 & 6 & 1 \\ 2 & -6 & 3 & 1 \\ 2 & 0 & -9 & -2 \\ -4 & 6 & 6 & 1 \end{bmatrix}$$

4. Write down $P_2, U_2$ and $X_3^{(b)}$.

$$P_2 = \begin{bmatrix} \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \\ & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} \end{bmatrix} \quad U_2 = \begin{bmatrix} \binom{1}{0} & \binom{1}{1} & \\ & \binom{1}{0} & \binom{1}{1} \end{bmatrix}$$

$$X_3^{(b)} = \begin{bmatrix} x & -(1-x) & & \\ & x & -(1-x) & \\ & & x & -(1-x) \end{bmatrix}$$

5. Calculate

$$P_2 \cdot N(F,G) = \begin{bmatrix} 2 & -6 & 3 & 1 \\ 2 & 0 & -9 & -2 \end{bmatrix}$$

$$U_2 \cdot X_3^{(b)} = \begin{bmatrix} x & -(1-x)+x & -(1-x) & \\ & x & -(1-x)+x & -(1-x) \end{bmatrix}$$

6. Therefore, by the Main Theorem

$$\begin{aligned} S_2(F,G) &= c \cdot \det \begin{bmatrix} P_2 \cdot N(F,G) \\ U_2 \cdot X_3^{(b)} \end{bmatrix} \\ &= \det \begin{bmatrix} 2 & -6 & 3 & 1 \\ 2 & 0 & -9 & -2 \\ x & -(1-x)+x & -(1-x) & \\ & x & -(1-x)+x & -(1-x) \end{bmatrix} \\ &= 150 w_{2,0}(x) - 150 w_{2,1}(x) - 75 w_{2,2}(x) \end{aligned}$$

# Bézout Subresultants of Bernstein Polynomials

## Corollary

Let $B^{(b)}(F,G)$ be the Bézout resultant matrix of $F$ and $G$ in Bernstein basis. Then for $k<n$,

$$S_k(F,G) = c \cdot \det \begin{bmatrix} P_k D_{m-1} B^{(b)}(F,1)^{-1} B^{(b)}(F,G) D_{m-1}^{-1} \\ U_k X_{m-1}^{(b)} \end{bmatrix}$$

where $c$, $P_k$, $U_k$ and $X_{m-1}^{(b)}$ as in the Main Theorem, and

$$D_{m-1} = \operatorname{diag} \begin{bmatrix} \frac{1}{\binom{m-1}{0}} & \frac{1}{\binom{m-1}{1}} & \cdots & \frac{1}{\binom{m-1}{m-1}} \end{bmatrix}$$

# An Illustrative Example

Consider $k = 2$ and

$$F(x) = 2\,w_{4,0}(x) - \frac{1}{2}\,w_{4,1}(x) - \frac{1}{2}\,w_{4,2}(x) - \frac{5}{4}\,w_{4,3}(x) - w_{4,4}(x)$$
$$G(x) = 4\,w_{3,0}(x) - 2\,w_{3,1}(x) - 2\,w_{3,2}(x) - \,w_{3,3}(x)$$

1. Recall $c$, $P_2$, $X_3^{(b)}$ and $U_1$ as in the previous example.
2. Construct $B^{(b)}(F,1)$ and $B^{(b)}(F,G)$.

$$B^{(b)}(F,1) = \begin{bmatrix} -10 & -5 & -\frac{13}{3} & -3 \\ -5 & -\frac{13}{9} & -\frac{5}{3} & -\frac{2}{3} \\ -\frac{13}{3} & -\frac{5}{3} & -\frac{20}{9} & -1 \\ -3 & -\frac{2}{3} & -1 & 1 \end{bmatrix} \quad B^{(b)}(F,G) = \begin{bmatrix} -4 & 4 & -2 & -2 \\ 4 & -\frac{8}{3} & -\frac{2}{3} & 0 \\ -2 & -\frac{2}{3} & \frac{13}{3} & 3 \\ -2 & 0 & 3 & 2 \end{bmatrix}$$

# An Illustrative Example (Cont.d)

3. Construct $D_3$.

$$D_3 = \begin{bmatrix} \frac{1}{\binom{3}{0}} & & & \\ & \frac{1}{\binom{3}{1}} & & \\ & & \frac{1}{\binom{3}{2}} & \\ & & & \frac{1}{\binom{3}{3}} \end{bmatrix}$$

4. Compute

$$P_2 D_3 B^{(b)}(F,1)^{-1} B^{(b)}(F,G) D_3^{-1} = \begin{bmatrix} 2 & -6 & 3 & 1 \\ 2 & 0 & -9 & -2 \end{bmatrix}$$

$$U_2 \cdot X_3^{(b)} = \begin{bmatrix} x & -(1-x)+x & -(1-x) & \\ & x & -(1-x)+x & -(1-x) \end{bmatrix}$$

5. By the Corollary

$$S_2(F,G) = c \cdot \det \begin{bmatrix} P_2 D_3 B(F,1)^{-1} B(F,G) D_3^{-1} \\ U_2 X_3^{(b)} \end{bmatrix}$$

$$= \det \begin{bmatrix} 2 & -6 & 3 & 1 \\ 2 & 0 & -9 & -2 \\ x & -(1-x)+x & -(1-x) & \\ & x & -(1-x)+x & -(1-x) \end{bmatrix}$$

$$= 150 w_{2,0}(x) - 150 w_{2,1}(x) - 75 w_{2,2}(x)$$

# Outline

# Subresultants in roots in standard basis

## Lemma (Hong et al. 99')

*We have*

$$S_k = c \cdot \frac{\det \begin{bmatrix} \boldsymbol{x}_{m-k-1}(\boldsymbol{\alpha}) \; G(\boldsymbol{\alpha}) \\ \boldsymbol{x}_{k-1}(\boldsymbol{\alpha}) \; (\boldsymbol{x}-\boldsymbol{\alpha}) \end{bmatrix}}{\det \boldsymbol{x}_{m-1}(\boldsymbol{\alpha})}$$

*where*

- $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, *and* $\alpha_1, \ldots, \alpha_m$ *are the roots of* $F$
- $\boldsymbol{x}_i = \begin{bmatrix} x^0, & x^1, & \ldots, & x^i \end{bmatrix}^T$
- $\boldsymbol{x}_i(\alpha_j) = \begin{bmatrix} \alpha_j^0, & \alpha_j^1, & \ldots, & \alpha_j^i \end{bmatrix}^T$
- $\boldsymbol{x}_i(\boldsymbol{\alpha}) = \begin{bmatrix} \boldsymbol{x}_i(\alpha_1), & \ldots, & \boldsymbol{x}_i(\alpha_m) \end{bmatrix}$
- $G(\boldsymbol{\alpha}) = \operatorname{diag} \begin{bmatrix} G(\alpha_1), & \ldots, & G(\alpha_m) \end{bmatrix}$
- $\boldsymbol{x} - \boldsymbol{\alpha} = \operatorname{diag} \begin{bmatrix} x - \alpha_1, & \ldots, & x - \alpha_m \end{bmatrix}$

# An Illustrative Example

Consider $k = 2$ and

$$F(x) = f_4 x^4 + f_3 x^3 + f_2 x^2 + f_1 x^1 + f_0 x^0$$
$$G(x) = g_3^3 + g_2 x^2 + g_1 x^1 + g_0 x^0$$

Let $\alpha_1, \ldots, \alpha_4$ be the roots of $F$. Then

$$S_2(F,G) = f_4 \cdot \frac{\det \begin{bmatrix} \alpha_1^0 G(\alpha_1) & \alpha_2^0 G(\alpha_2) & \alpha_3^0 G(\alpha_3) & \alpha_4^0 G(\alpha_4) \\ \alpha_1^1 G(\alpha_1) & \alpha_2^1 G(\alpha_2) & \alpha_3^1 G(\alpha_3) & \alpha_4^1 G(\alpha_4) \\ \alpha_1^0 (x-\alpha_1) & \alpha_2^0 (x-\alpha_2) & \alpha_3^0 (x-\alpha_3) & \alpha_4^0 (x-\alpha_4) \\ \alpha_1^1 (x-\alpha_1) & \alpha_2^1 (x-\alpha_2) & \alpha_3^1 (x-\alpha_3) & \alpha_4^0 (x-\alpha_4) \end{bmatrix}}{\det \begin{bmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 & \alpha_4^0 \\ \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & \alpha_4^1 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \end{bmatrix}}$$

# Derivation

**Step 1.** Deduce the equivalent expression **in terms of roots** in scaled Bernstein basis.

**Lemma (Subresultants in roots in scaled Bernstein basis )**

*We have*

$$S_k(F,G) = c \cdot \frac{\det \begin{bmatrix} \overline{\boldsymbol{w}}_{m-k-1}(\boldsymbol{\alpha}) G(\boldsymbol{\alpha}) \\ \overline{\boldsymbol{w}}_{k-1}(\boldsymbol{\alpha})(\boldsymbol{x} - \boldsymbol{\alpha}) \end{bmatrix}}{\det \ \overline{\boldsymbol{w}}_{m-1}(\boldsymbol{\alpha})}$$

*where*

- $\overline{\boldsymbol{w}}_i(\boldsymbol{\alpha}) = \begin{bmatrix} \overline{\boldsymbol{w}}_i(\alpha_1), & \ldots, & \overline{\boldsymbol{w}}_i(\alpha_m) \end{bmatrix}$
- $\overline{\boldsymbol{w}}_i(\alpha_j) = \begin{bmatrix} \overline{w}_{i,0}(\alpha_j), & \ldots, & \overline{w}_{i,i}(\alpha_j) \end{bmatrix}^T$

# An Illustrative Example

Consider $k = 2$ and

$$F(x) = a_0 w_{4,0}(x) + a_1 w_{4,1}(x) + a_2 w_{4,2}(x) + a_3 w_{4,3}(x) + a_4 w_{4,4}(x)$$
$$G(x) = b_0 w_{3,0}(x) + b_1 w_{3,1}(x) + b_2 w_{3,2}(x) + b_3 w_{3,3}(x)$$

Let $\alpha_1, \ldots, \alpha_4$ be the roots of $F$. Then

$$S_2(F,G) = f_4 \cdot \frac{\det \begin{bmatrix} \overline{w}_{1,0}(\alpha_1)G(\alpha_1) & \overline{w}_{1,0}(\alpha_2)G(\alpha_2) & \overline{w}_{1,0}(\alpha_3)G(\alpha_3) & \overline{w}_{1,0}(\alpha_4)G(\alpha_4) \\ \overline{w}_{1,1}(\alpha_1)G(\alpha_1) & \overline{w}_{1,1}(\alpha_2)G(\alpha_2) & \overline{w}_{1,1}(\alpha_3)G(\alpha 3) & \overline{w}_{1,1}(\alpha_4)G(\alpha_4) \\ \overline{w}_{1,0}(\alpha_1)(x-\alpha_1) & \overline{w}_{1,0}(\alpha_2)(x-\alpha_2) & \overline{w}_{1,0}(\alpha_3)(x-\alpha_3) & \overline{w}_{1,0}(\alpha_4)(x-\alpha_4) \\ \overline{w}_{1,1}(\alpha_1)(x-\alpha_1) & \overline{w}_{1,1}(\alpha_2)(x-\alpha_2) & \overline{w}_{1,1}(\alpha_3)(x-\alpha_3) & \overline{w}_{1,1}(\alpha_4)(x-\alpha_4) \end{bmatrix}}{\det \begin{bmatrix} \overline{w}_{3,0}(\alpha_1) & \overline{w}_{3,0}(\alpha_2) & \overline{w}_{3,0}(\alpha_3) & \overline{w}_{3,0}(\alpha_4) \\ \overline{w}_{3,1}(\alpha_1) & \overline{w}_{3,1}(\alpha_2) & \overline{w}_{3,1}(\alpha_3) & \overline{w}_{3,1}(\alpha_4) \\ \overline{w}_{3,2}(\alpha_1) & \overline{w}_{3,2}(\alpha_2) & \overline{w}_{3,2}(\alpha_3) & \overline{w}_{3,2}(\alpha_4) \\ \overline{w}_{3,3}(\alpha_1) & \overline{w}_{3,3}(\alpha_2) & \overline{w}_{3,3}(\alpha_3) & \overline{w}_{3,3}(\alpha_4) \end{bmatrix}}$$

# Derivation

**Step 2.** Prove that the companion subresultants are equivalent to the subresultants in roots in **scaled Bernstein basis**:

$$\det \begin{bmatrix} P_k N(F,G) \\ U_k X_{m-1}^{(b)} \end{bmatrix} = \frac{\det \begin{bmatrix} \overline{\boldsymbol{w}}_{m-k-1}(\boldsymbol{\alpha}) G(\boldsymbol{\alpha}) \\ \overline{\boldsymbol{w}}_{k-1}(\boldsymbol{\alpha})(\boldsymbol{x}-\boldsymbol{\alpha}) \end{bmatrix}}{\det \ \overline{\boldsymbol{w}}_{m-1}(\boldsymbol{\alpha})}$$

**Step 3.** Using the connection between $N(F,G)$ and $B^{(b)}(F,G)$, we deduce $k$-th **Bézout subresultant**

$$B^{(b)}(F,G) = B^{(b)}(F,1) D_{m-1}^{-1} N(F,G) D_{m-1}$$

# Outline

# Conclusion

## Summary

Provide two types of subresultant formulas for Bernstein polynomials

(1) **Companion subresultant**

(2) **Bézout subresultant**

## Future work

(1) **Computation: e.g., develop fast algorithms**

(2) **Application: e.g., intersections of two Bezier curves**

(3) **Generalization: e.g., multiple polynomials and multivariate cases**

# Thank you for your attention!