Considerations on Approaches and Metrics in Automated Theorem Generation/Finding in Geometry

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Problem 31

> *What properties can be identified to permit an automated reasoning program to find new and interesting theorems, as opposed to proving conjectured theorems?*
>
> *Automated Reasoning: 33 Basic Research Problems*, *Larry Wos*

Two (big!!!) problems in a single (small) sentence:

- discover new theorems;
- select interesting theorems.

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deductive application of sound inference rules to axioms

+ sound
− avoid the generation of uninteresting consequences

The Deductive Approach

New Consequences are generated by application of sound inference rules to the axioms and previously generated logical consequences.

This can be done by an appropriately configured saturation-based ATP system.

The advantage of this approach is that only logical consequences are ever generated.

The challenge of this approach is to avoid the many uninteresting logical consequences that can be generated.

Strong Relevant Logic-based Forward Deduction Approach

Jingde Cheng Cheng [2000] claims that classical mathematical logic, its various classical conservative extensions, and traditional (weak) relevant logics cannot satisfactorily underlie epistemic processes in scientific discovery, presenting an approach based on strong relevant logic.

Hongbiao Gao et al. have followed this approach applying it for several domains such as NBG set theory, Tarski's Geometry and Peano's Arithmetic Gao and Cheng [2017], Gao et al. [2014, 2018], Gao et al. [2019]

Rule Based Systems

One of the ATP built-in in *JGEx* is an implementation of the geometry deductive database method Chou et al. [2000], Ye et al. [2011]. Using a breadth-first forward chaining a fix-point for the conjecture at hand is reached.

The geometry deductive database method proceeds by using a simple algorithm where, starting from the geometric construction D_0, the rules, R, are applied over and over till a fix-point, D_k is reached:

$$\boxed{D_0} \overset{R}{\subset} \boxed{D_1} \overset{R}{\subset} \cdots \overset{R}{\subset} \boxed{D_k} \quad \text{(fix-point)}$$

A new open source implementation of this method, *OGP-GDDM*,* is described in Baeta and Quaresma [2023]

*https://github.com/opengeometryprover/OpenGeometryProver

Algebraic e Approaches

A similar approach is taken in the well-known dynamic geometry system *GeoGebra*.[†]

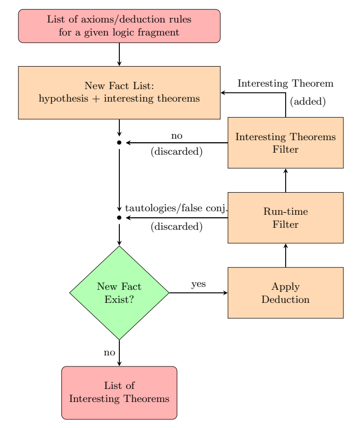
The *GeoGebra Discovery* reports some facts that were systematically checked from a list of possible features including identical points, parallel or perpendicular lines, equal long segments, collinearity or concyclicity.

This is not a deductive method so the generation process must be externally verified.

[†] https://www.geogebra.org/

Automated Theorem Finding (ATF)

The Deductive Approach Algorithm



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The filtering for interesting theorems or for uninteresting conjectures, two sides of the same coin, is done by application of a series of filters.

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Usefulness: measures how much an interesting theorem has contributed to proofs of further interesting theorems.

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Undecidability Result

Theorem (Undecidability Result)

For any given Turing Machine, it is undecidable to determine, whether the language recognised by it has the property of finding interesting theorems.

By application of the Rice's Theorem that, *The problem of determining whether a given Turing machine's language has a non-trivial property p is undecidable*, after having stated that *p, the property that can establish if a given theorem is interesting, is a non-trivial property*.

It is undecidable to have a deterministic program that can find interesting problems. At best this is a task to be addressed by programs based on algorithms guided by heuristics criteria.

Designing Interesting Surveys

In light of our undecidability result, to understand what experts mean by, "a program that is able to also prove interesting theorems", must be done referring to empirical data, via the formulation of an expert survey.

One has to first reach a minimal degree of agreement on the definition of interestingness of theorems.

In order to achieve this agreement, an empirical exploration of the notion of interestingness and of what it concretely entails is paramount.

What is Interestingness

We will ask the experts:

- Some situations in which they remember to have used the adjective interesting concerning a theorem, and to explain the use of this expression.
- list several geometric theorems they find interesting
- list several geometric theorems they find not interesting,

Levels of Interestingness

We will ask these experts whether they find the theorems listed interesting or not. We will ask them to rate, using a Likert scale.

Is Theorem n interesting?	Yes/No
Why? Because is has the characteristic $A/B/C/D$	
	Strongly disagree. . . Neutral. . . Strongly Agree

Design Issues

With a Little Help from My Friends.

The Beatles, Sgt. Peppers Lonely Hearts Club Band

With an agreement on what an interesting theorem is, we could query experts in theorem generators/finders design, with another survey (the third survey) asking how to design software able to produce these interesting theorems.

Obrigado Grazie

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