Euclid's theorems through the Area Method

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Problem and Motivation

Dual nature of the theory of similar figures

Ancient mathematics could process line segments, triangles, rectangles, and other geometric magnitudes and instead of real numbers and fractions employed proportions.

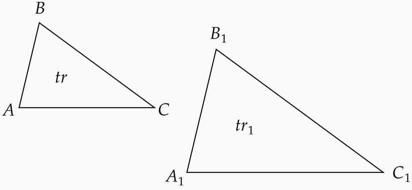
$$\frac{\triangle}{\triangle_1} = \frac{tr}{tr_1} = \frac{sq}{sq_1} = \frac{\square}{\square_1}$$

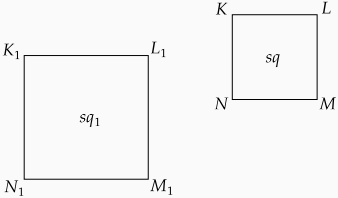
where $=$ stands for proportion, and $\triangle, \triangle_1, \square, \square_1$ are geometric figures

In contemporary school mathematics, fractions mimic proportions and are subject to operations within the arithmetic of real numbers.

$$\frac{a}{b} = \frac{S_{ABC}}{S_{A_1B_1C_1}} = \frac{S_{KLMN}}{S_{K_1L_1M_1N_1}} = \frac{c}{d}$$

where $a, b, c, d \in \mathbb{R}$, and S_{ABC}, etc. stand for areas of figures





The intercept (Thales) theorem

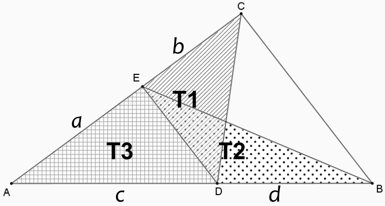
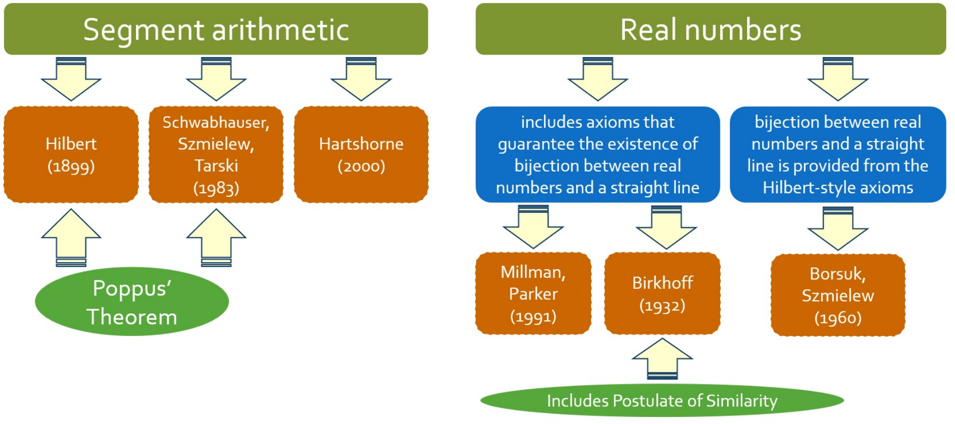


Figure 1: *Elements*, proposition VI.2

$$T1 = \triangle(CDE),\ \ T2 = \triangle(BDE), T3 = \triangle(ADE)$$
$$\frac{T2}{T3} = \frac{d}{c},\ \ \frac{T1}{T3} = \frac{b}{a},$$
$$DE \| BC \ then \ T2 = T1 \ (Prop.I.38),$$
$$T2 = T1 \ then \ \frac{T1}{T3} = \frac{T2}{T3} \ (Prop.V.7),$$
$$\frac{T2}{T3} = \frac{d}{c} = \frac{T1}{T3} = \frac{b}{a} = \frac{T1}{T3}$$

Thales' Theorem (20th century foundations of geometry)



State of the Art

Two ways of interpreting the Elements

The first originates in Hilbert's *Grundlagen der Geometrie*. It consists of a reconstruction of the logical structure of the treaty. Euclid's propositions derive from a system containing original axioms and new ones.

The second trend respects that Euclid refers to diagrams in many proofs. This way of interpreting elevates diagrams in mathematical argumentation.

The theory of proportions developed by Hilbert, and later by Hartshorne, can not reconstruct Euclid's theorems concerning proportions of figures, e.g., VI.1, 16, 19, 20, 31.In turn, authors who emphasize the role of diagrams cannot explain the comparison of figures in terms of greater-lesser when one figure does not contain the other.

Contribution and Main Idea

Euclid's proposition VI.1

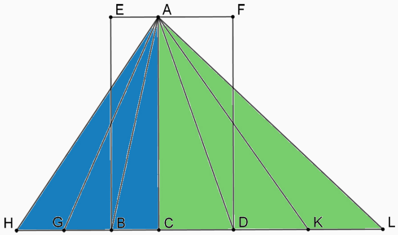


Figure 2: *Elements*, proposition VI.1

"Let ABC and ACD be triangles, [...] of the same height AC. I say that as base BC is to base CD, so triangle ABC (is) to triangle ACD".

We have a proportion $a : b :: c : d$. In the theorem VI.1 a, b are triangles, and c, d segments. We formalize it as follows $\triangle ABC : \triangle ADC :: BC : DC$.

Proof:

$BC = GB = HG$ and
$CD = DK = KL$

$\triangle AHC = 3\triangle ABC$, and
$\triangle ALC = 3\triangle ADC$

If $3\triangle ABC \gtreqqless 3\triangle ADC$, then $3BC \gtreqqless 3DC$.

Euclid's proposition VI.1

Euclid's proof is cumbersome, to say the least; it applies non-defined concepts of the addition of triangles encoded in the notion of *multiple* and requires comparing triangles in terms of *greater-lesser*. The accompanying diagram is to represent relations $\triangle AHC = 3\triangle ABC$, and $\triangle ALC = 3\triangle ADC$. Somehow, we must decide that $\triangle AHC < \triangle ALC$ whenever $HC < LC$.

In our reconstruction of Book VI, we eliminate deliberations of this kind by accepting it as an axiom.

The primitive concepts of the Area Method

The primitive concepts of the system of axioms of the Area Method are:

- the point $(A, B, C \ldots)$,
- the length of a directed segment $(\overline{AB})$
- the signed area of a triangle (S_{ABC}), which are elements of an ordered field.

$\overline{AB}$ and S_{ABC} can be positive, negative, or zero and processed in an ordered arithmetic field. These concepts allow us to interpret segments, triangles, and polygons without reference to real numbers.

Definition (1)
Points A, B, C are collinear iff $S_{ABC} = 0$.

Definition (2)
Two segments AD and BC, where $A \neq D$ and $B \neq C$, are parallel, iff $S_{ABC} = S_{DBC}$. We adopt the standard symbol $AD \parallel BC$ for this relation.

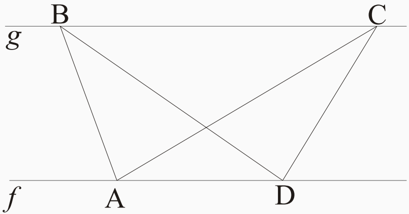


Figure 3: Definition of parallel line segments

Some axioms of the Area Method

A1. $\overline{AB} = 0$ if and only if A and B are identical.

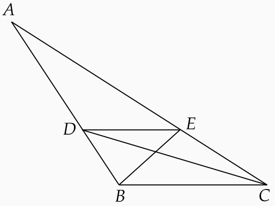
A2. $S_{ABC} = S_{CAB}$.

A3. $S_{ABC} = -S_{BAC}$.

A5. There are points A, B and C such that $S_{ABC} \neq 0$ (not all points are collinear).

A10. If $S_{PAC} \neq 0$ and $S_{ABC} = 0$, then $\frac{\overline{AB}}{\overline{AC}} = \frac{S_{PAB}}{S_{PAC}}$ (Euclid's proposition VI.1).

The proof of Thales' theorem in the Area Method



Euclid's proof

$$
\begin{aligned}
DE \| BC &\rightarrow \triangle DEB = \triangle DEC \\
\triangle DEB = \triangle DEC &\rightarrow \triangle DEB : \triangle DAE :: \triangle DEC : \triangle DAE \\
E \perp AB &\rightarrow \triangle DEB : \triangle DAE :: BD : DA \\
D \perp AC &\rightarrow \triangle DEC : \triangle DAE :: CE : EA \\
&\rightarrow BD : DA :: CE : EA
\end{aligned}
$$

note: $E \perp AB$ represents the the stipulation "having the same height" namely, the straight line drawn from E perpendicular to AB.

Area method proof

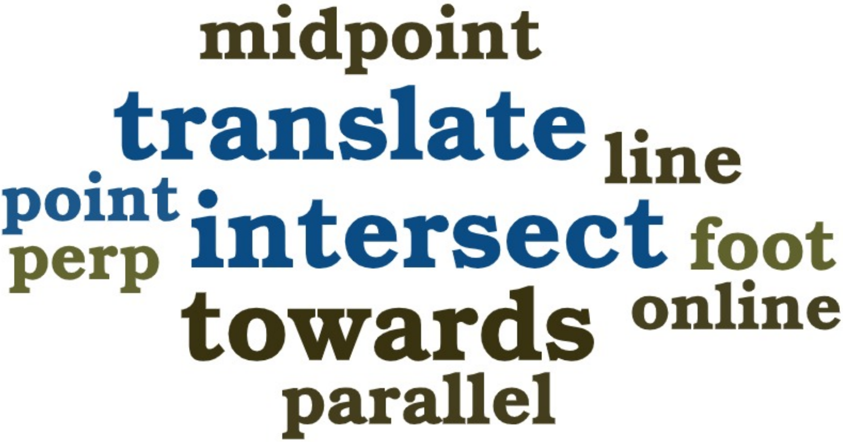
$$
\begin{aligned}
DE \| BC &\rightarrow S_{DEB} = S_{DEC} \\
S_{DEB} = S_{DEC} &\rightarrow \frac{S_{DEB}}{S_{DAE}} = \frac{S_{DEC}}{S_{DAE}} \\
S_{ADB} = 0 &\rightarrow \frac{S_{EBD}}{S_{EDA}} = \frac{\overline{BD}}{\overline{DA}} \\
S_{AEC} = 0 &\rightarrow \frac{S_{DCE}}{S_{DEA}} = \frac{\overline{CE}}{\overline{EA}} \\
&\rightarrow \frac{\overline{BD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EA}}
\end{aligned}
$$

Elimination lemmas

Elimination lemmas specify this procedure. Given that, an automated proof proceeds as follows:

1. The thesis of a theorem is translated into an expression in the Area Method language.
2. Given some starting points, new points are introduced, one by one, through the allowed constructions (construction stage).
3. Each point introduced in the construction stage is eliminated based on elimination lemmas, but in reverse order, i.e., the last constructed is the first in the elimination process, etc. (elimination stage).
4. The process reaches identity $1 = 1$ or $0 = 0$ and stops.

Allowed constructions in GCLC



Brief introduction to the used terminology

Table 1: Geometry quantities in GCLC

ratio of directed segments	$\frac{\overline{PQ}}{\overline{AB}}$	`sratio P Q A B`
signed area (arity 3)	S_{ABC}	`signed_area3 A B C`

Table 2: Statements for the basic sorts of conjectures in GCLC

points A and B are identical	`identical A B`
points A, B, C are collinear	`collinear A B C`
AB is perpendicular to CD	`perpendicular A B C D`
AB is parallel to CD	`parallel A B C D`
O is the midpoint of AB	`midpoint O A B`
AB has the same length as CD	`same_length A B C D`

Theorem VI.2, construction at GCLC

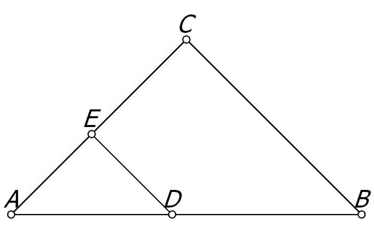


Figure 4:
Theorem VI.2, construction at GCLC

Construction steps:

```
point A 20 30
point B 60 30
point C 40 50
online D A B
line bc B C
line ca C A
parallel de D bc
intersec E ca de
```

Theorem thesis in terms of automatic proof:

```
prove{ equal{ sratio B D D A }{ sratio C E E A }}}
```

In ordinary mathematical language, it can be written like this:

$$\frac{\overline{BD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EA}}.$$

The fragment of the proof generated by the GCLC

$\frac{\overrightarrow{BD}}{\overrightarrow{DA}}=\frac{\overrightarrow{CE}}{\overrightarrow{EA}}$	by the statement
$\frac{\overrightarrow{BD}}{\overrightarrow{DA}}=\left(-1\cdot\frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)$	by geometric simplifications
$\frac{\overrightarrow{BD}}{\overrightarrow{DA}}=\left(-1\cdot\frac{S_{CDP^1_{de}}}{S_{ADP^1_{de}}}\right)$	by Lemma 8 (point E eliminated)
$\frac{\overrightarrow{BD}}{\overrightarrow{DA}}=\frac{\left(-1\cdot S_{CDP^1_{de}}\right)}{S_{ADP^1_{de}}}$	by algebraic simplifications
$\cdots$	
$(r_0+(-1\cdot(r_0\cdot r_0)))=\left(\frac{(0+r_0)}{\frac{\overrightarrow{AB}}{\overrightarrow{AB}}}+\left(-1\cdot\left(\frac{\overrightarrow{AD}}{\overrightarrow{AB}}\cdot r_0\right)\right)\right)$	by geometric simplifications
$r_0=\frac{\overrightarrow{AD}}{\overrightarrow{AB}}$	by algebraic simplifications
$r_0=\frac{\left(\frac{\overrightarrow{AA}}{\overrightarrow{AB}}+r_0\right)}{\frac{\overrightarrow{AB}}{\overrightarrow{AB}}}$	by Lemma 39 (point D eliminated)
$r_0=\frac{(0+r_0)}{\frac{\overrightarrow{AB}}{\overrightarrow{AB}}}$	by geometric simplifications
$0=0$	by algebraic simplifications $0=0$

Theorem (VI.3)

Let ABC be a triangle. Let the angle BAC be cut in half by the straight line AD. I say that as BD is to CD, so BA is to AC.

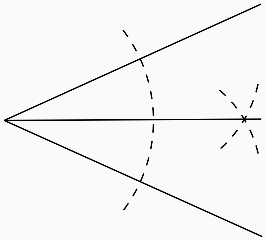


Figure 5: Construction of angle bisector

Constructing a bisector in GCLC

New task: constructing a bisector using legal commands in GCLC-prover.

We cannot construct a bisector in a defined angle, but we can construct a "some" angle with the bisector.

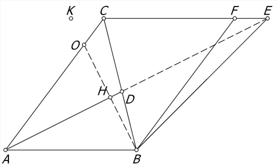


Figure 6: Theorem VI.3, construction at GCLC

```
point A 40 30
point B 70 30
point K 55 60
line kb K B
foot H A kb
translate O B H H
online C O A
line ah A H
line bc B C
intersec D ah bc
line ab A B
parallel cd C ab
line ac A C
parallel bf B ac
intersec E ah cd
intersec F bf cd
```

Constructing a bisector in GCLC

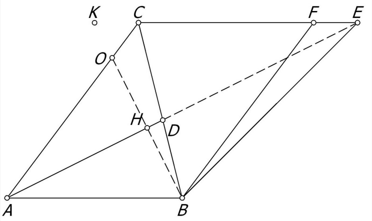


Figure 7: Theorem VI.3, construction at GCLC

Let us have any points A, B, and K. Construct a perpendicular AH from point A to the segment BK. Then, we construct a point O symmetrical to B with respect to H (the construction command `translate`). Triangle BAO is isosceles and AH is a bisector of $\angle A$. Choose any point C on line AO.

The point D is the intersection of AH and BC, and AD is the bisector of angle CAB. The ratio command can only be used on parallel segments. In the case of AB and AC, there are no parallel segments. We make additional constructs: $ABFC$ is a parallelogram $\Rightarrow AB = CF$, triangle ACE is isosceles $\Rightarrow CA = CE$.

Finally, we can formulate the thesis in terms of the GCLC:

```
prove{ equal { sratio B D C D } { sratio  F C C E}}
```

Case for bisector of outside angle

Above, we have discussed the case where AD is the bisector of an inside angle of the triangle, but in GCLC [7], we immediately get the theorem for the case of the bisector of the triangle's outside angle, too (see Fig. 8).

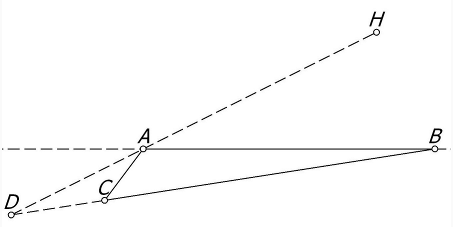


Figure 8: Theorem VI.3 [6, p. 158], construction at GCLC [7] (case for bisector of outside angle)

Conclusion

Conclusion

- The Area Method and Prover GCLC enable automatic proofs of propositions from Book VI of the Elements.
- The proposed method enables the reconstruction of Euclid's theses and the original proof technique, i.e., the proportion theory.
- Just as constructions are the crux of Euclid's proofs, understanding automatic proofs reduces to elimination lemmas.
- These lemmas refer to Euclid's technique of constructing points while adding a new aspect to that process, namely the elimination.
- Since mechanical proofs are crucial to modern mathematics, we seek to introduce this method into the teaching process.
- Algorithms used in Euclidean geometry, i.e., the Area Method, are well suited for teaching. In this way, ancient mathematics supported by new technology will introduce students to a brand new mathematical idea, namely automatic proof.

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