The locus story of a rocking camel in a medical center in the city of Freistadt

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ADG 2023, September 20, 2023, Belgrade, Serbia

Third author is supported by the grant PID2020-113192GB-I00
from the Spanish MICINN.

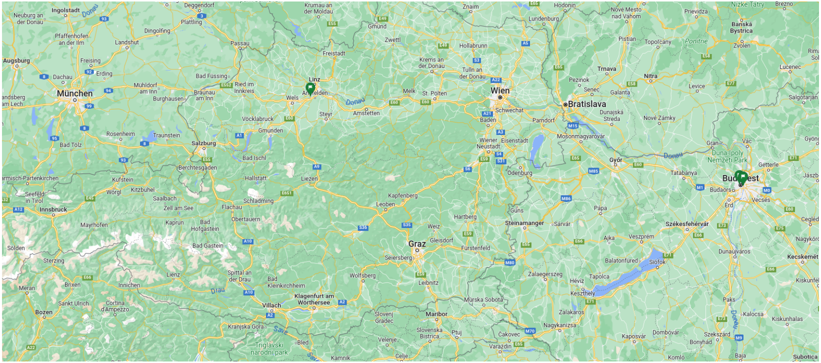
Abstract

We give an example of automated geometry reasoning for an imaginary classroom project by using the free software package *GeoGebra Discovery*.

The project is motivated by a publicly available toy, a rocking camel, installed at a medical center in Upper Austria. We explain how the process of

- a false conjecture,
- experimenting,
- modeling,
- a precise mathematical setup,
- and then a proof by automated reasoning

could help extend mathematical knowledge at secondary school level and above.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Freistadt
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Pezinok
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Linz
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Ried im Innkreis
Senec
Galanta
Levice
München
Braunau am Inn
Burghausen
Wels
Ansfelden
Melk
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Wien
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Steyr
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Liezen
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Budapest
Kufstein
Budaörs
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Kitzbühel
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Vecsés
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Kiskunfélegyháza
Keszthely
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Murska Sobota
Paks
Sankt Ulrich
Cortina d'Ampezzo
Villach
Bozen
Slovenj Gradec
Maribor
Zalakaros
Kalocsa
Kiskunhalas
Kranjska Gora
Slovenska Bistrica
Ptuj
Čakovec
Nagykanizsa
Kaposvár
Dombóvár
Szekszárd
Velenje
Bled
Triglavski narodni park
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ROTSCHOPF
freistädter
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RATSHERRN
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Movement of the hump of the camel

Video recordings → static images → conjecture

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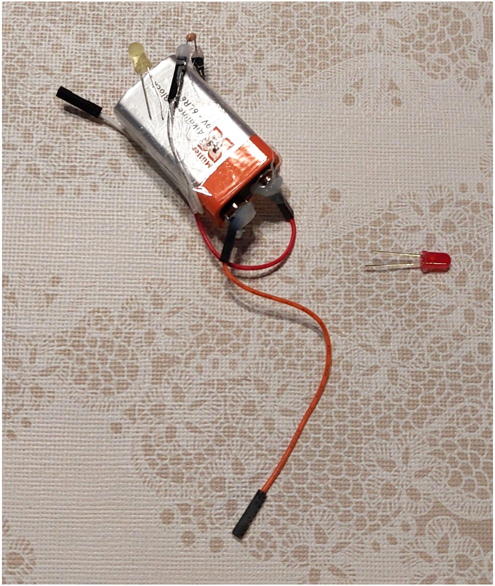

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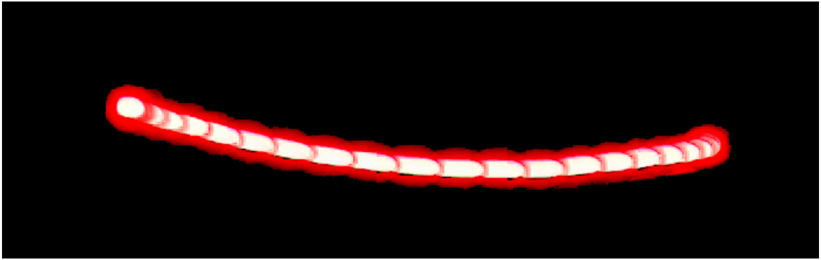

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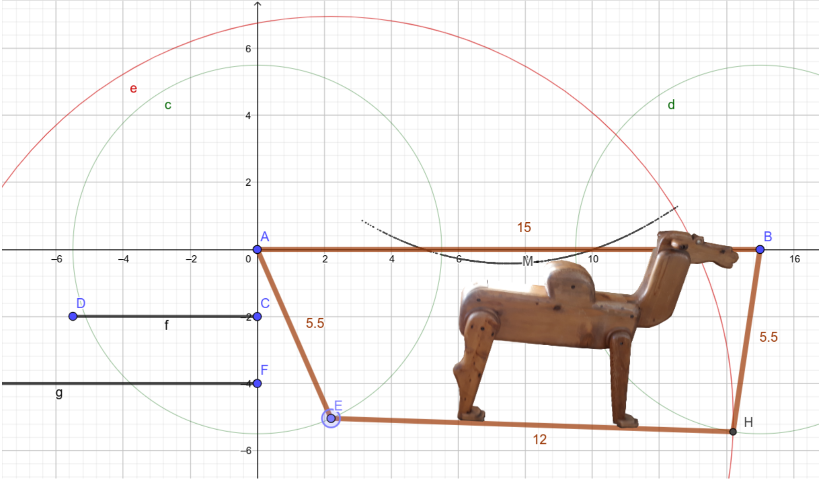

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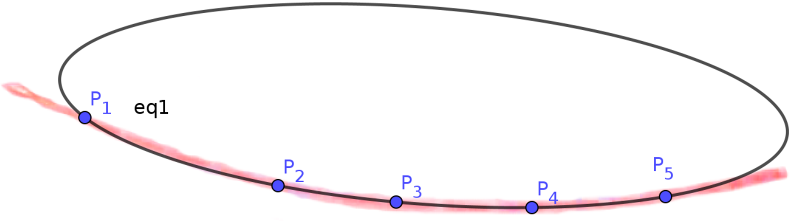

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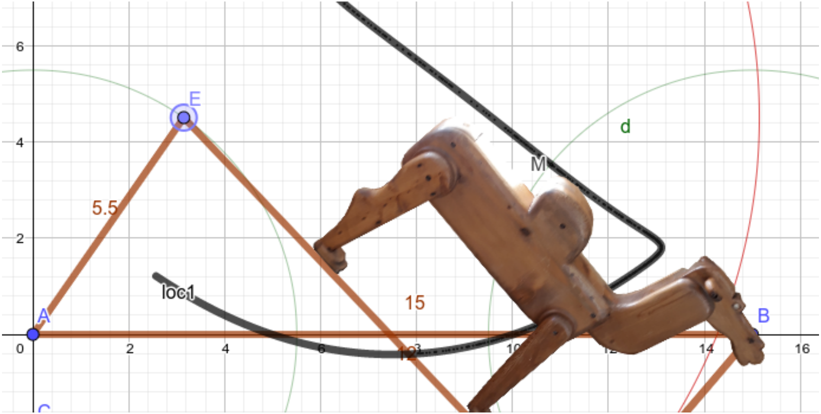

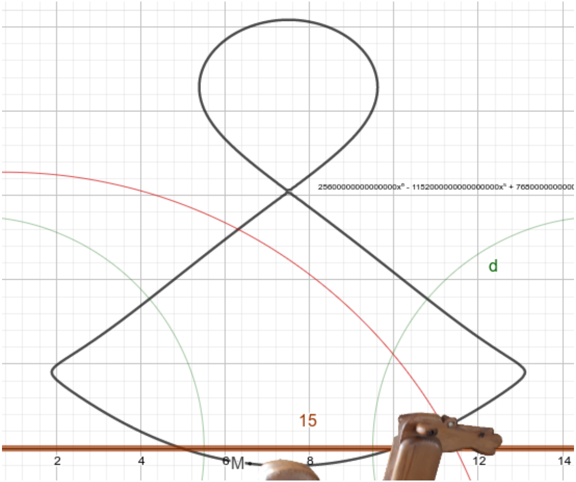

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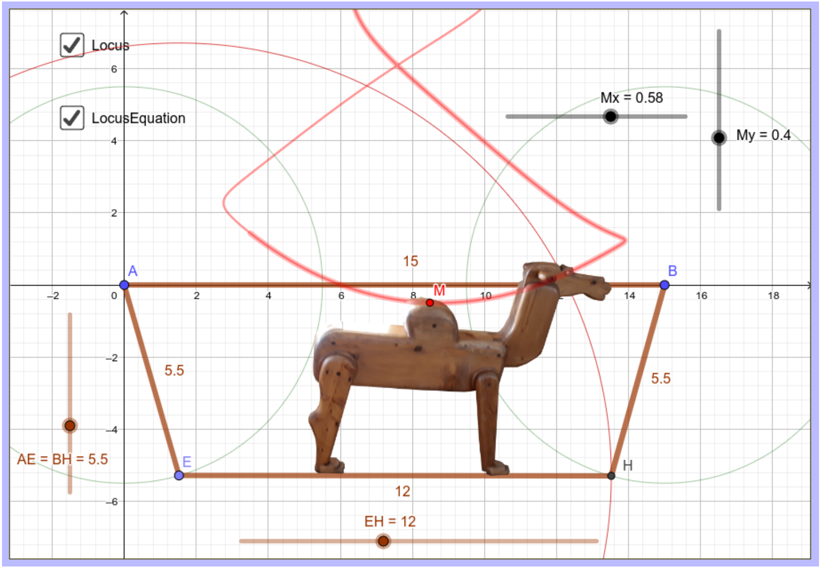
Movement of the hump of the camel
Computing the locus equation



$$625000000x^6 - 29625000000x^5 + 1875000000x^4y^2 - 7500000000x^4y + 513916750000x^4 - 59250000000x^3y^2 + 225000000000x^3y - 3894242700000x^3 + 1875000000x^2y^4 - 15000000000x^2y^3 + 701583500000x^2y^2 - 2494326000000x^2y + 12634068729100x^2 - 29625000000xy^4 + 225000000000xy^3 - 3894242700000xy^2 + 14694390000000xy - 26440635548340x + 625000000y^6 - 7500000000y^5 + 187666750000y^4 - 2494326000000y^3 + 23089046979100y^2 - 75203840809200y = -80422746144129$$

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6. $\langle a^2 + b^2 - 5.5^2, (c - 15)^2 + d^2 - 5.5^2,$
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9. Try to generalize the problem with different inputs. (Difficult!)

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Further uses of the approach

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   Here we get a quadratic result — in general, this can be much more complicated. (Collinearity: degree 1 — but: conchoids, cissoids, strophoids (of degree 3)
   or cardioids, deltoids or lemniscates (of degree 4)
   were mostly quite well-known
   by the ancient Greek mathematicians!)

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