Towards automated readable proofs of ruler and compass constructions

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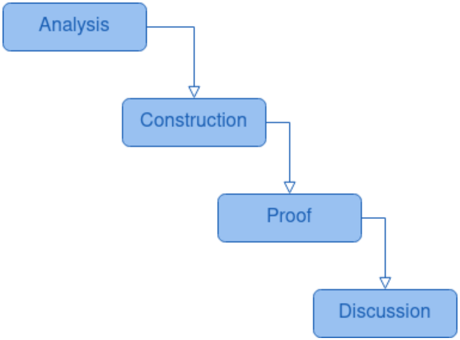
Solving ruler and compass construction problems

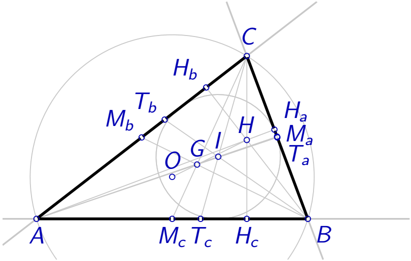
- One of the most studied problems in mathematical education
- Task: to describe a construction of geometrical figure which satisfies given set of constraints
  " construct $\triangle ABC$ given α, β and $|AB|$"
- Constructions are procedures
- Some instances are unsolvable (e.g. angle trisection)

Phases in solving construction problems



ArgoTriCS

- ArgoTriCS – system for automated solving of location construction problems from the given corpus (authors: V. Marinković, P. Janičić)
- Task of location triangle construction problem is to construct $\triangle ABC$ if locations of three significant points in the triangle are given
- Tool was tested on Wernick's corpus



- Requires background geometrical knowledge

ArgoTriCS

1. *Using the point A and the point H_a, construct a line h_a (rule W02);*

   % DET: points A and H_a are not the same

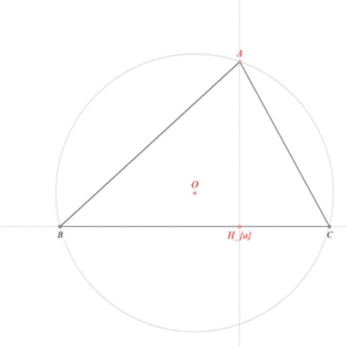
2. *Using the point A and the point O, construct a circle k(O,C) (rule W06);*

   % NDG: points A and O are not the same

3. *Using the point H_a and the line h_a, construct a line a (rule W10a);*

4. *Using the circle k(O,C) and the line a, construct a point C and a point B (rule W04);*

   % NDG: line a and circle k(O,C) intersect



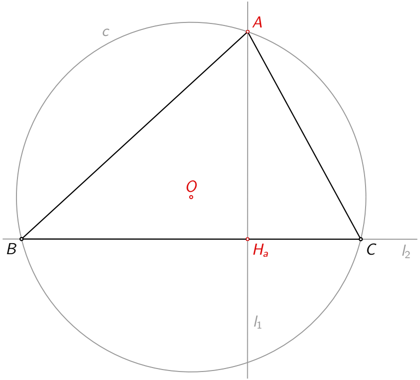
- Exports informal textual description of construction, as well as formal description of construction in GCLC and JSON format
- Enables generation of dynamic illustrations
- Constructions are proved correct using algebraic and semi-algebraic methods

The goal of research

- Existing systems for solving RC-constructions DO NOT provide classical, human-readable synthetic correctness proofs
- In current work we propose first steps towards obtaining readable, but also formal correctness proofs of automatically generated RC-constructions
- Synergy of various tools: triangle construction solver ArgoTriCS, FOL provers, coherent logic provers and interactive theorem provers

Example 1 – construction phase

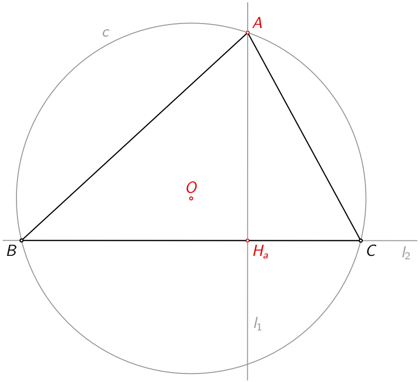
- Task: Construct $\triangle ABC$ given its vertex A, circumcenter O, and altitude foot H_a



1. Construct the line $l_1 = AH_a$
2. Construct the line $l_2 \colon l_2 \perp l_1$ and $H_a \in l_2$
3. Construct the circle c centered at O containing A
4. Let B and C be the intersections of the line l_2 and the circle c

Example 1 – proof phase

- Task: Prove that A is the vertex of the constructed triangle ABC, that H_a is its altitude foot and that O is its circumcenter



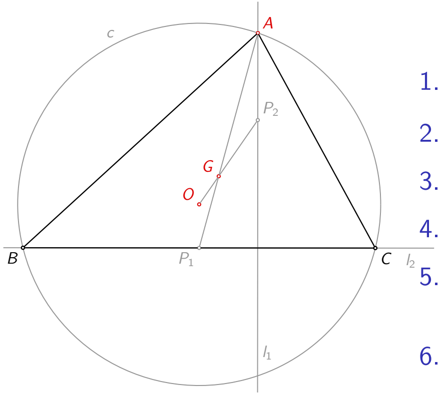
1. c contains vertices A, B, and C, so it must be the circumcircle of $\triangle ABC$
2. O is the center of c, so it must be the circumcenter of $\triangle ABC$
3. l_2 contains the vertices B and C, so it must be equal to side a of $\triangle ABC$
4. l_1 contains A and is perpendicular to $l_2 = a$, so it must be equal to altitude h_a
5. H_a belongs both to $l_2 = a$ and $l_1 = h_a$, so it must be the altitude foot

Conclusions following from Example 1

- The previous correctness proof follows quite directly from the analysis: it just reverses the chain of deduction steps
- The proof relies on several uniqueness lemmas
- One could conclude that it is always easy like this, however...
- ... in some cases the proof is quite different from the analysis

Example 2 – construction phase

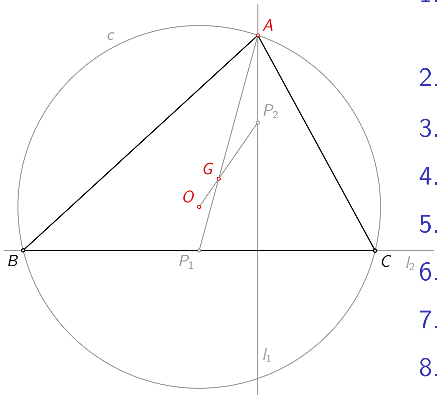
- Task: Construct $\triangle ABC$ given its vertex A, circumcenter O and centroid G



1. Construct the point $P_1 : \overrightarrow{AG} : \overrightarrow{AP_1} = 2 : 3$
2. Construct the point $P_2 : \overrightarrow{OG} : \overrightarrow{OP_2} = 1 : 3$
3. Construct the line $l_1 = AP_2$
4. Construct the line $l_2 : l_2 \perp l_1$ and $P_1 \in l_2$
5. Construct the circle c centered at O containing A
6. Let B and C be the intersections of the line l_2 and the circle c

Example 2 – proof phase

- Task: Prove that A is the vertex of the constructed triangle ABC, that G is its centroid and that O is its circumcenter



1. ... similarly to earlier we get that O is the circumcenter of $\triangle ABC$, $l_2 = a$ and $l_1 = h_a$
2. $\overrightarrow{OG} : \overrightarrow{OP_2} = 1 : 3 \Rightarrow \overrightarrow{OG} : \overrightarrow{GP_2} = 1 : 2$
3. $\overrightarrow{P_1G} : \overrightarrow{P_1A} = 1 : 3 \Rightarrow \overrightarrow{P_1G} : \overrightarrow{GA} = 1 : 2$
4. Triangles OGP_1 and P_2GA are similar
5. Angles $\angle OP_1G$ and $\angle GAP_2$ are equal
6. Lines $OP_1 = l_3$ and $AP_2 = h_a$ are parallel
7. $h_a \perp a \Rightarrow l_3 \perp a$
8. l_3 is perpendicular bisector of BC
9. $P_1 = M_a$
10. $\overrightarrow{AG} : \overrightarrow{AM_a} = 2 : 3 \Rightarrow G$ is centroid of $\triangle ABC$

Automated generation of readable correctness proofs

- How can correctness proofs like the ones we have seen be automatically obtained?
- We need to formulate the problem statement and the set of lemmas, given as axioms and to pass them to some automated theorem prover

Problem statement

- ArgoTriCS can automatically generate the theorem (in a form suitable for ATPs) stating that the generated construction is correct

$$
\begin{aligned}
&\mathrm{inc}(A, l_1) \wedge \mathrm{inc}(H'_a, l_1) \wedge \\
&\mathrm{perp}(l_2, l_1) \wedge \mathrm{inc}(H'_a, l_2) \wedge \\
&\mathrm{circle}(O', A, c) \wedge \\
&\mathrm{inc}(B, l_2) \wedge \mathrm{inc}(C, l_2) \wedge \mathrm{inc\_c}(B, c) \wedge \mathrm{inc\_c}(C, c) \wedge B \neq C \implies \\
&H'_a = H_a \wedge O' = O
\end{aligned}
$$

- H'_a and O' are the points given, while H_a and O are the real altitude foot and circumcenter of constructed triangle ABC
- Various non-degeneracy conditions are added to the problem statement (e.g., $H'_a \neq A$, $A \neq B$, $A \neq C$, etc.) before it is given to ATPs

The axiom set for proof phase

- Definitions and lemmas identified by ArgoTriCS

$$\begin{aligned} \mathrm{inc}(A, h_a) &\wedge \mathrm{perp}(h_a, bc) \\ \overrightarrow{AG} : \overrightarrow{AM_a} &= 2 : 3 \end{aligned}$$

- Uniqueness lemmas

$$\begin{aligned} (\forall l)(\mathrm{inc}(A, l) \wedge \mathrm{perp}(l, bc) &\implies l = h_a) \\ (\forall c)(\mathrm{inc\_c}(A, c) \wedge \mathrm{inc\_c}(B, c) \wedge \mathrm{inc\_c}(C, c) &\implies c = c^\circ) \end{aligned}$$

- Properties of basic geometry predicates

$$\begin{aligned} (\forall l_1, l_2)(\mathrm{perp}(l_1, l_2) &\implies \mathrm{perp}(l_2, l_1)) \\ (\forall P_1, P_2)(\exists l)(\mathrm{inc}(P_1, l) &\wedge \mathrm{inc}(P_2, l)) \\ (\forall l1, l2, a)\ (\mathrm{perp}(l_1, a) \wedge \mathrm{para}(l_1, l_2) &\implies \mathrm{perp}(l_2, a)) \end{aligned}$$

Using automated theorem provers

- Problem statement and identified lemmas are formulated in TPTP format
- The conjecture is passed to automated theorem prover Vampire and coherent logic prover Larus
- Vampire is much more efficient, but Larus exports both readable proofs and formal proofs

Example of readable correctness proof

Axioms:

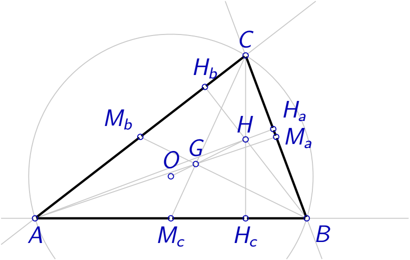
1. bc_unique : $\forall L\ (inc(pB, L) \wedge inc(pC, L) \Rightarrow L = bc\ )$
2. haA : $\forall H\ (perp(H, bc) \wedge inc(pA, H) \Rightarrow ha = H\ )$
3. pHa_def : $\forall H1\ (inc(H1, ha) \wedge inc(H1, bc) \Rightarrow H1 = pHa\ )$
4. cc_unique : $\forall C\ (inc\_c(pA, C) \wedge inc\_c(pB, C) \wedge inc\_c(pC, C) \Rightarrow C = cc\ )$
5. center_unique : $\forall C\ \forall C1\ \forall C2\ (center(C1, C) \wedge center(C2, C) \Rightarrow C1 = C2\ )$

Theorem: th_A_Ha_O0 :
$inc(pA, ha1) \wedge inc(pHa1, ha1) \wedge perp(ha1, a1) \wedge inc(pHa1, a1) \wedge inc\_c(pA, cc1) \wedge center(pOc1, cc1) \wedge inc\_c(pB, cc1) \wedge inc(pB, a1) \wedge inc\_c(pC, cc1) \wedge inc(pC, a1) \Rightarrow pHa = pHa1$
Proof:

1. $pHa = pHa$ (by MP, using axiom eqnativeEqSub0; instantiation: $A \mapsto pHa$, $B \mapsto pHa$, $X \mapsto pHa$)
2. $a1 = bc$ (by MP, from $inc(pB, a1)$, $inc(pC, a1)$ using axiom bc_unique; instantiation: $L \mapsto a1$)
3. $perp(ha1, bc)$ (by MP, from $perp(ha1, a1)$, $a1 = bc$ using axiom perpEqSub1; instantiation: $A \mapsto ha1$, $B \mapsto a1$, $X \mapsto bc$)
4. $ha = ha1$ (by MP, from $perp(ha1, bc)$, $inc(pA, ha1)$ using axiom haA; instantiation: $H \mapsto ha1$)
5. $inc(pHa1, ha)$ (by MP, from $inc(pHa1, ha1)$, $ha = ha1$ using axiom incEqSub1; instantiation: $A \mapsto pHa1$, $B \mapsto ha1$, $X \mapsto ha$)
6. $inc(pHa1, bc)$ (by MP, from $inc(pHa1, a1)$, $a1 = bc$ using axiom incEqSub1; instantiation: $A \mapsto pHa1$, $B \mapsto a1$, $X \mapsto bc$)
7. $pHa1 = pHa$ (by MP, from $inc(pHa1, ha)$, $inc(pHa1, bc)$ using axiom pHa_def; instantiation: $H1 \mapsto pHa1$)
8. $pHa = pHa1$ (by MP, from $pHa1 = pHa$, $pHa = pHa$ using axiom eqnativeEqSub0; instantiation: $A \mapsto pHa$, $B \mapsto pHa1$, $X \mapsto pHa$)
9. Proved by assumption! (by QEDas)

Results

- The subset of problems from Wernick's corpus is considered: 35 non-isomorphic solvable location triangle problems over
  - vertices A, B, C
  - side midpoints M_a, M_b, M_c
  - feet of altitudes H_a, H_b, H_c
  - centroid G, circumcenter O and orthocenter H



- Vampire succesfully proved 31 problem
- Larus successfully proved 20 problems within the given time-limit of 300 seconds

Conclusions

- Work-in-progress
- First step toward automated readable, synthetic, formally verified correctness proofs
- Important for educational purposes
- Lemmas identified during development of ArgoTriCS were needed, but they were not sufficient
- Coherent logic provers are still not as efficient as automated theorem provers

Future work

- Proofs currently rely on high-level lemmas
- Correctness of used lemmas should be proved: we are currently developing formal Isabelle/HOL proofs for all lemmas from the basic geometric axioms
- We plan to consider degenerate cases and existence of constructed objects
- We plan to exploit concept of hints avaliable in Larus, to help it prove some more conjectures