Using GXWeb for theorem proving and mathematical modelling

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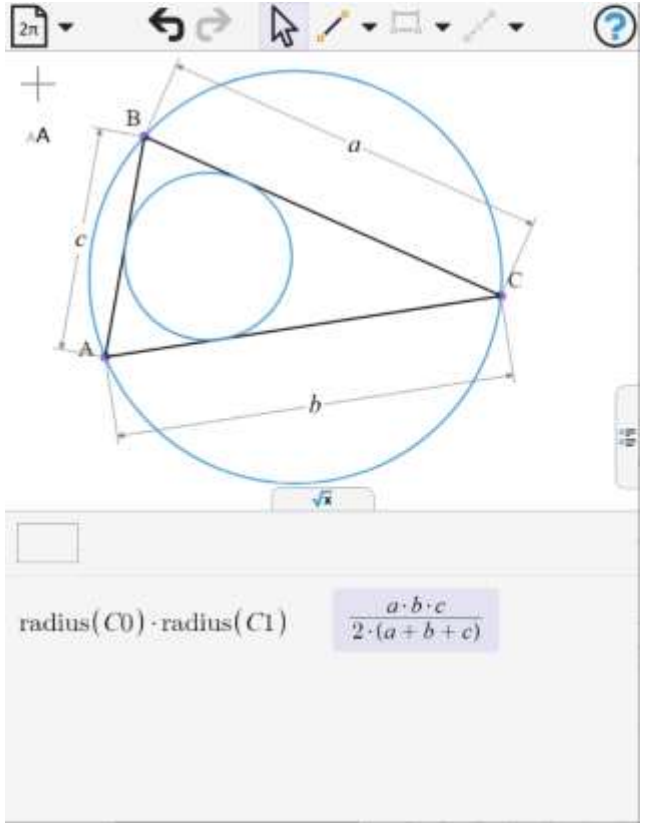
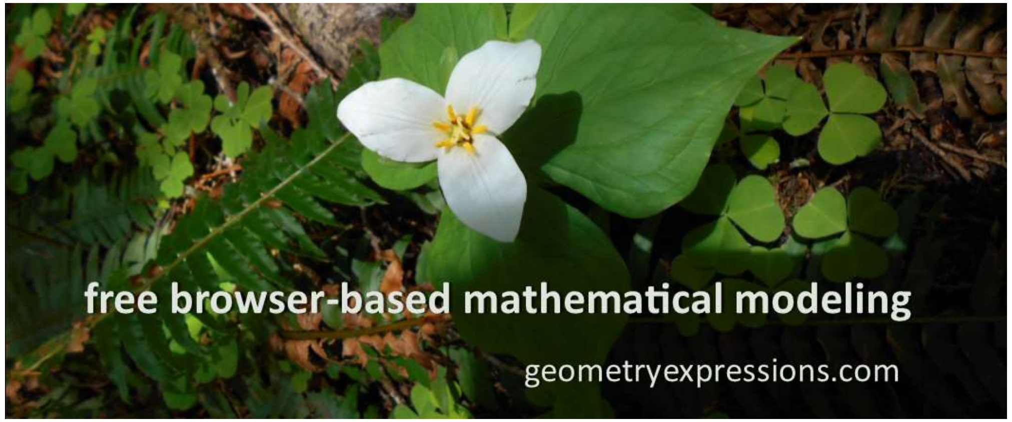




History

2006

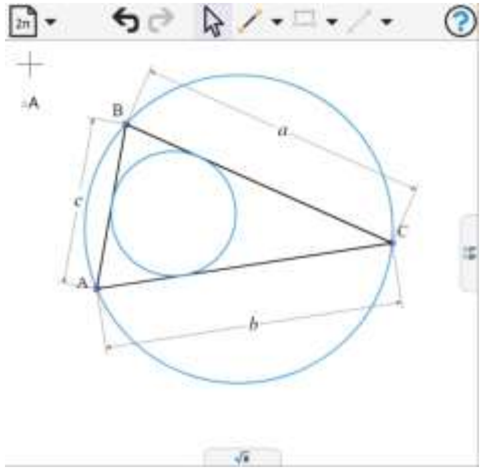
Symbolic geometry, available on Mac or PC.
Commercial (non-free) product.

2017

Free browser based version of Geometry Expressions

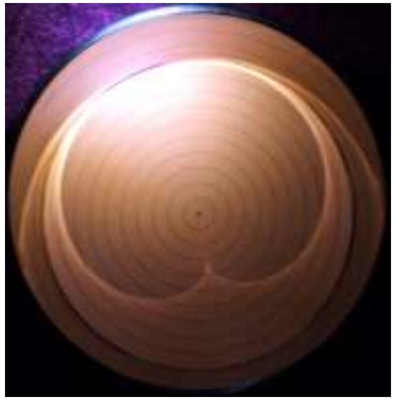
Context

We demonstrate the software in the following three contexts

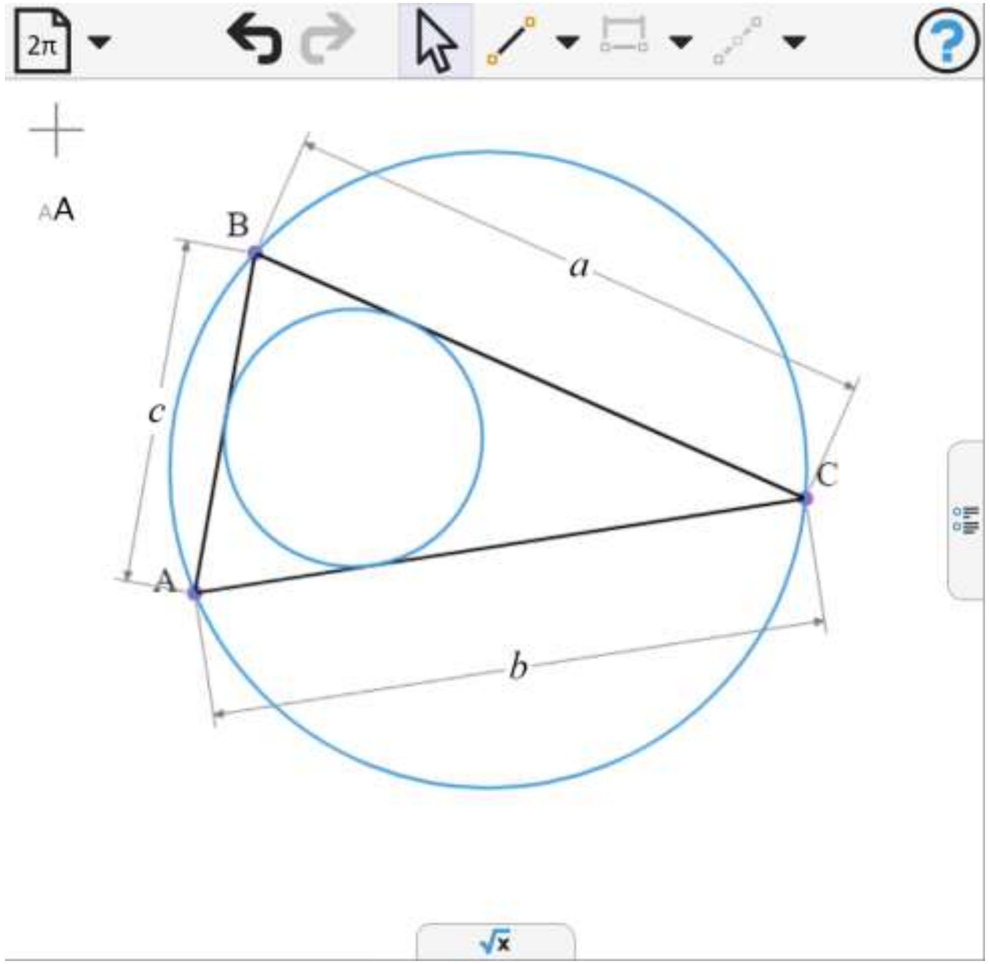


Incircle radius – geometry theorem proving

Box solar cooker – mathematical modelling

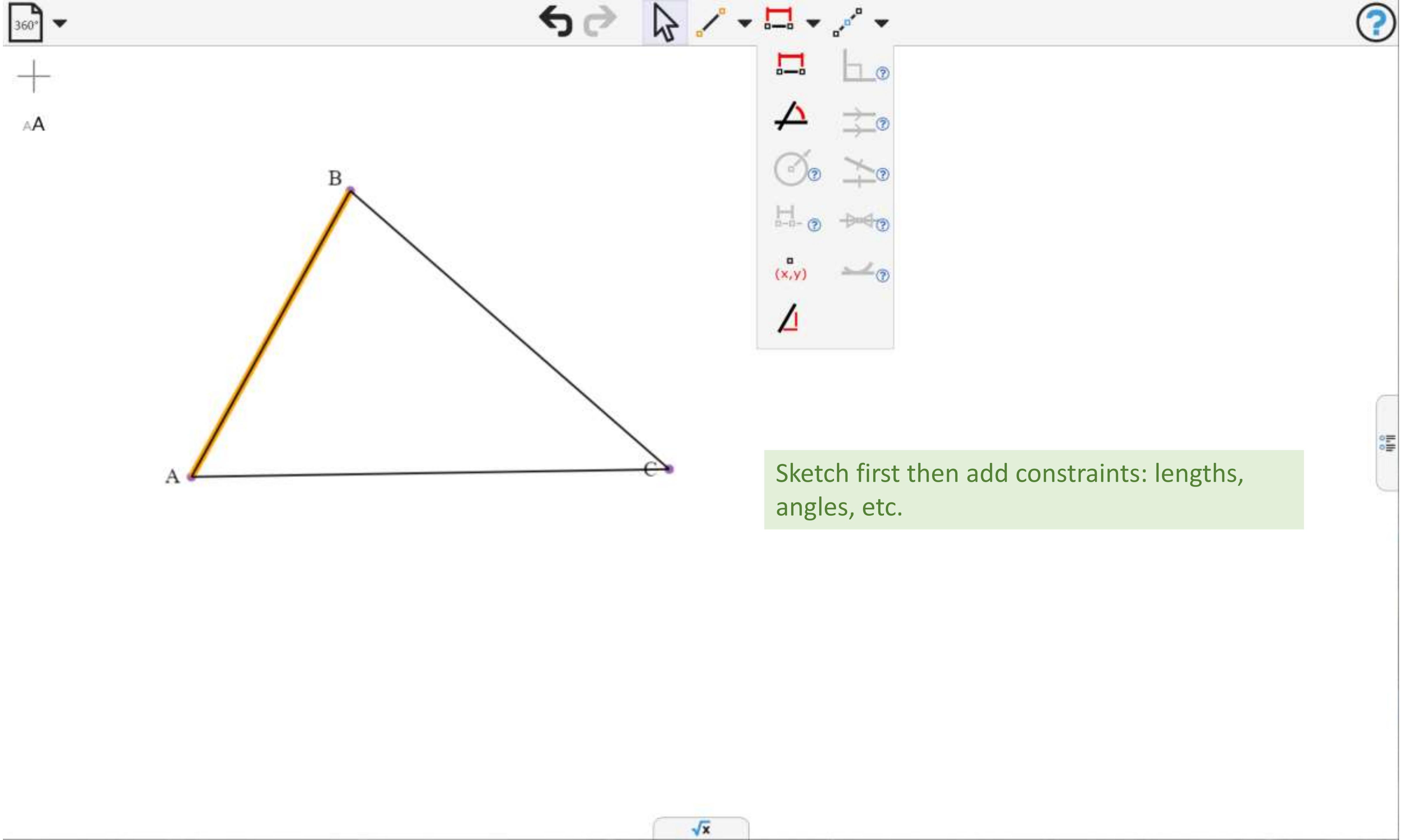


Circle caustics – loci and envelopes

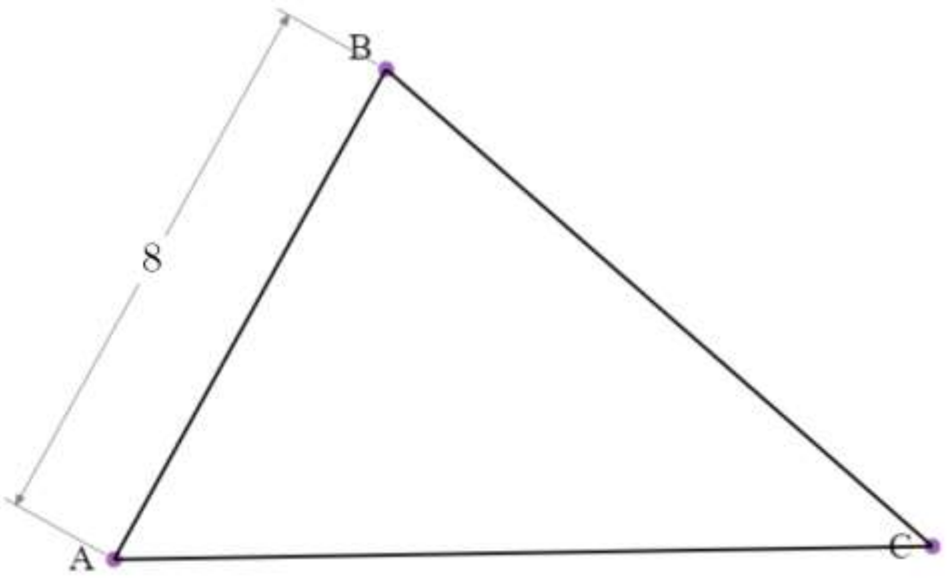


Incircle radius – theorem proving

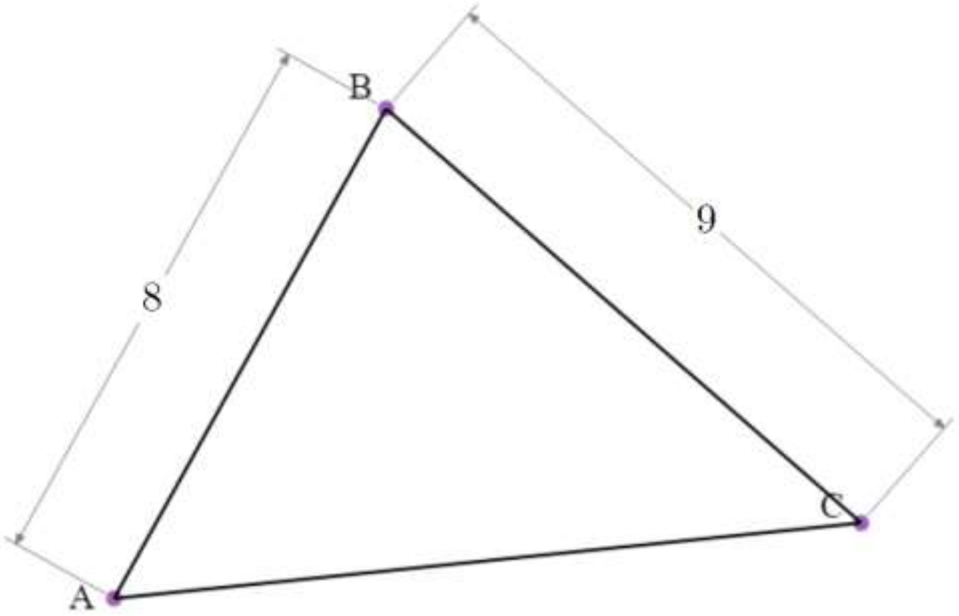
Geometry



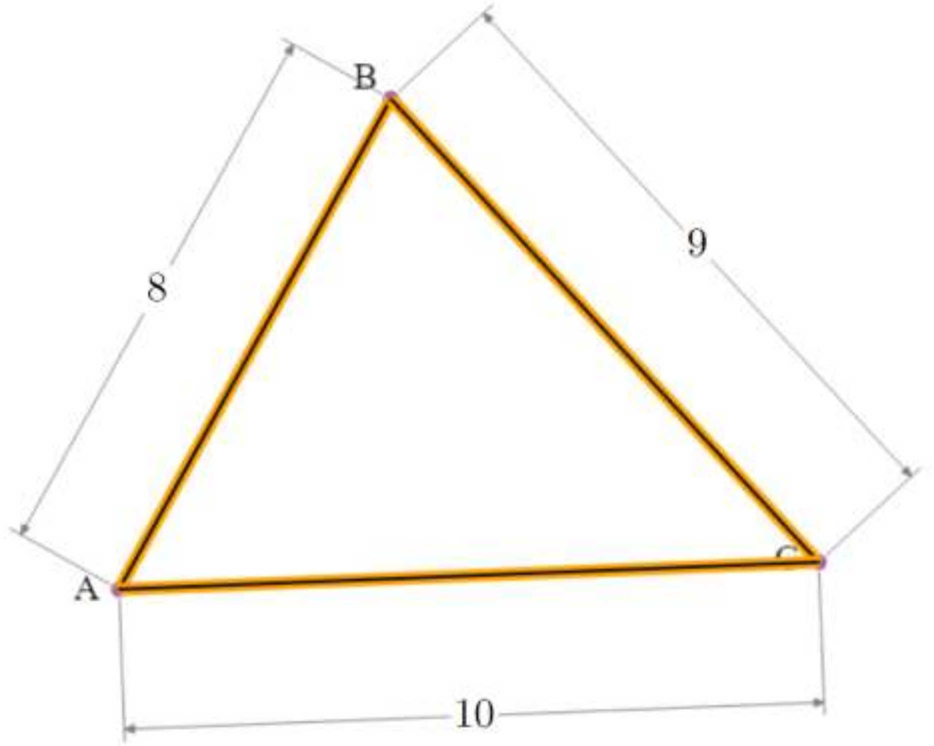
Sketch first then add constraints: lengths, angles, etc.



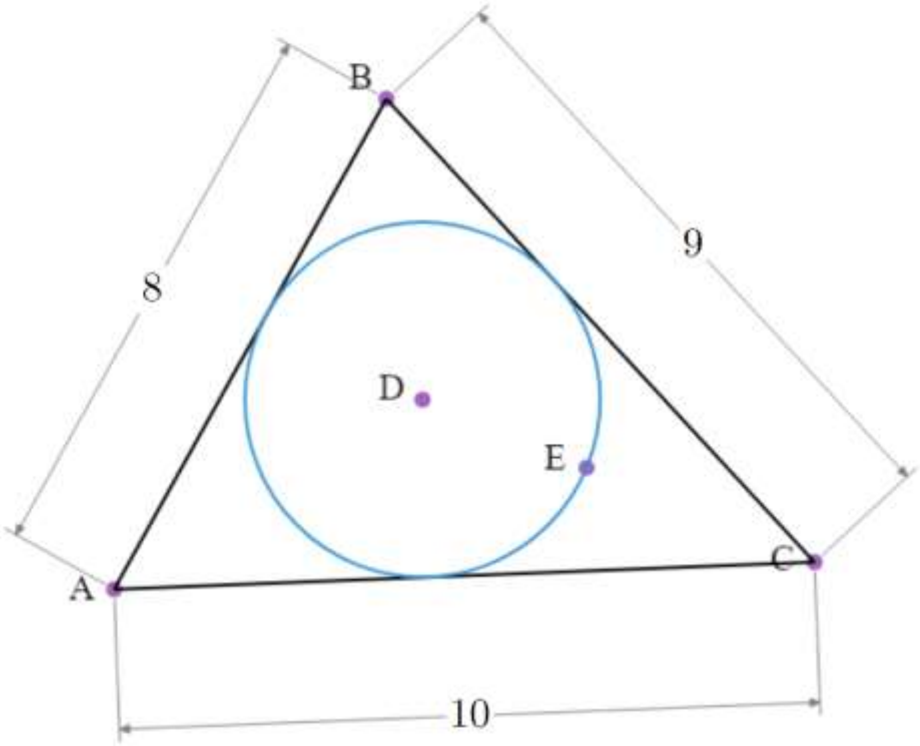
First length just rescales the model, so no visible change.



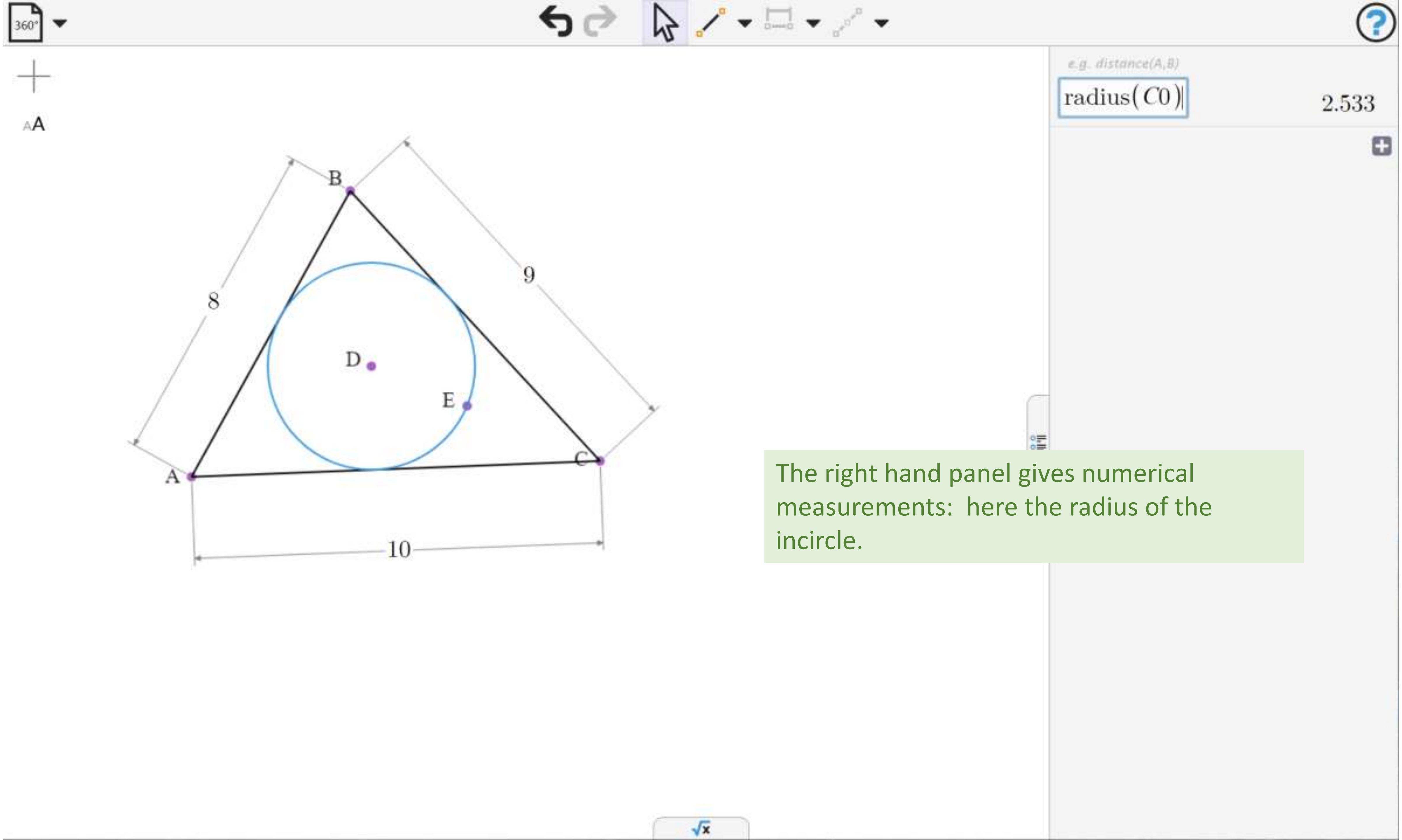
Next length makes a difference

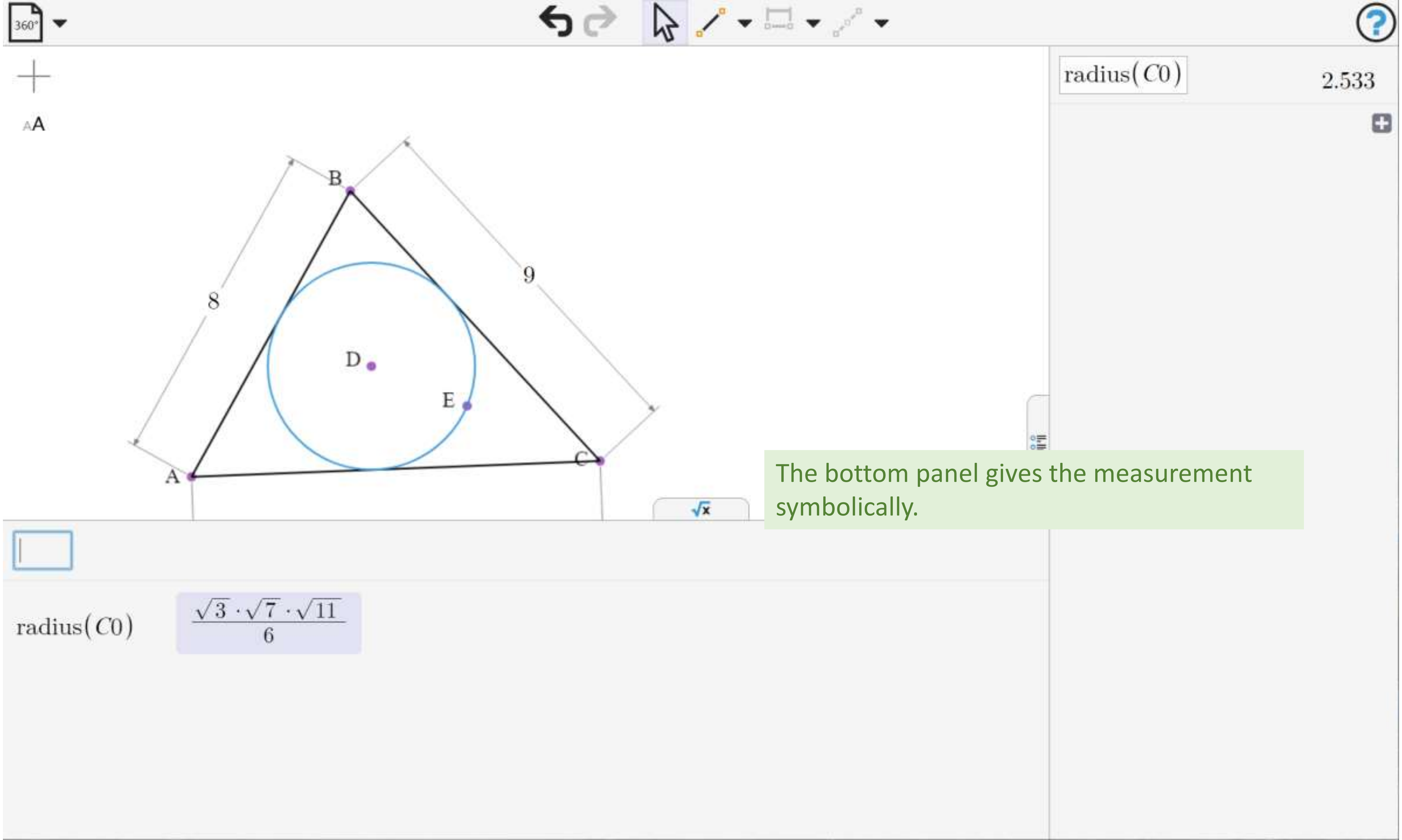


To create the incircle, we select the three sides, than use the circle construction

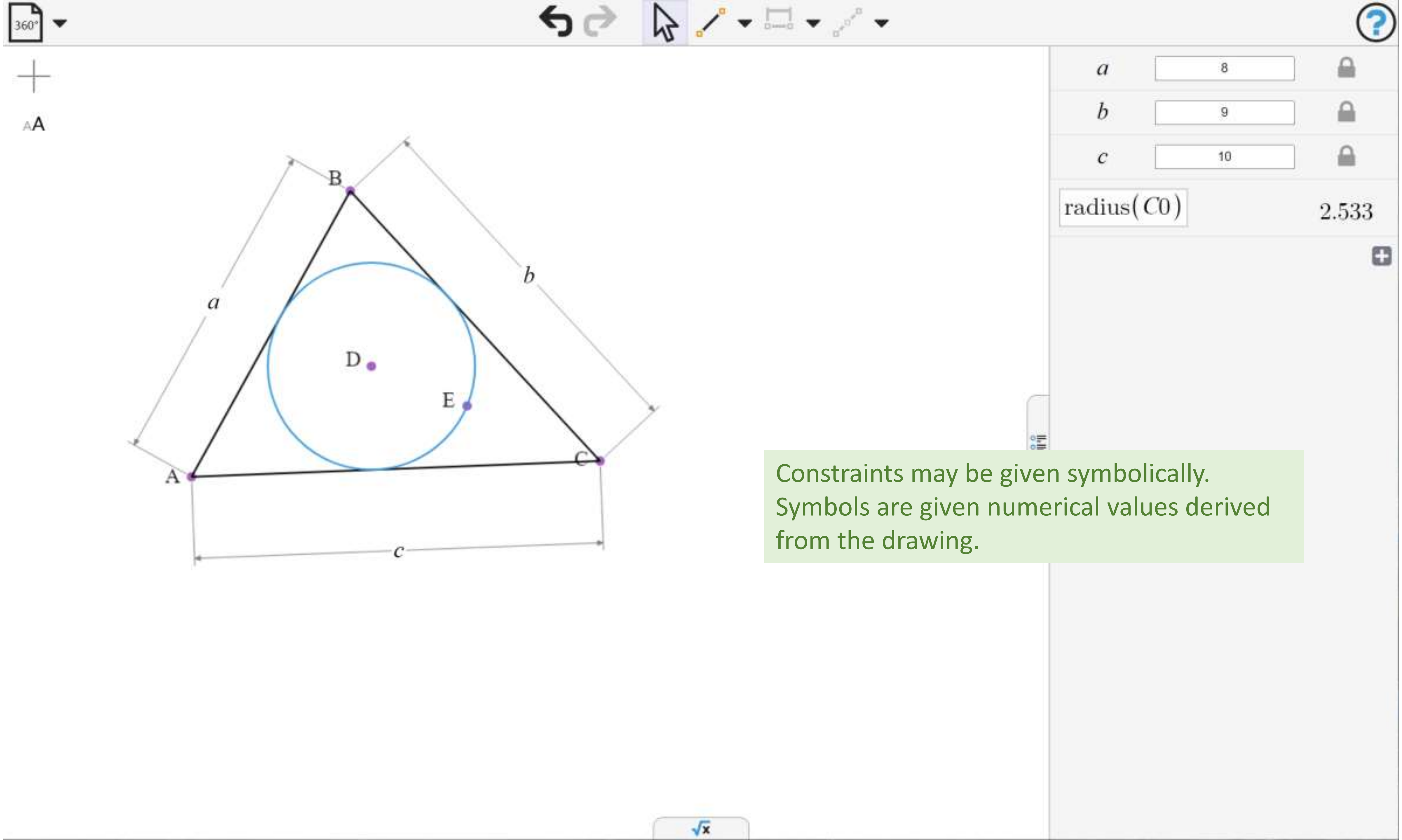


B
8
9
D
E
A
C
10

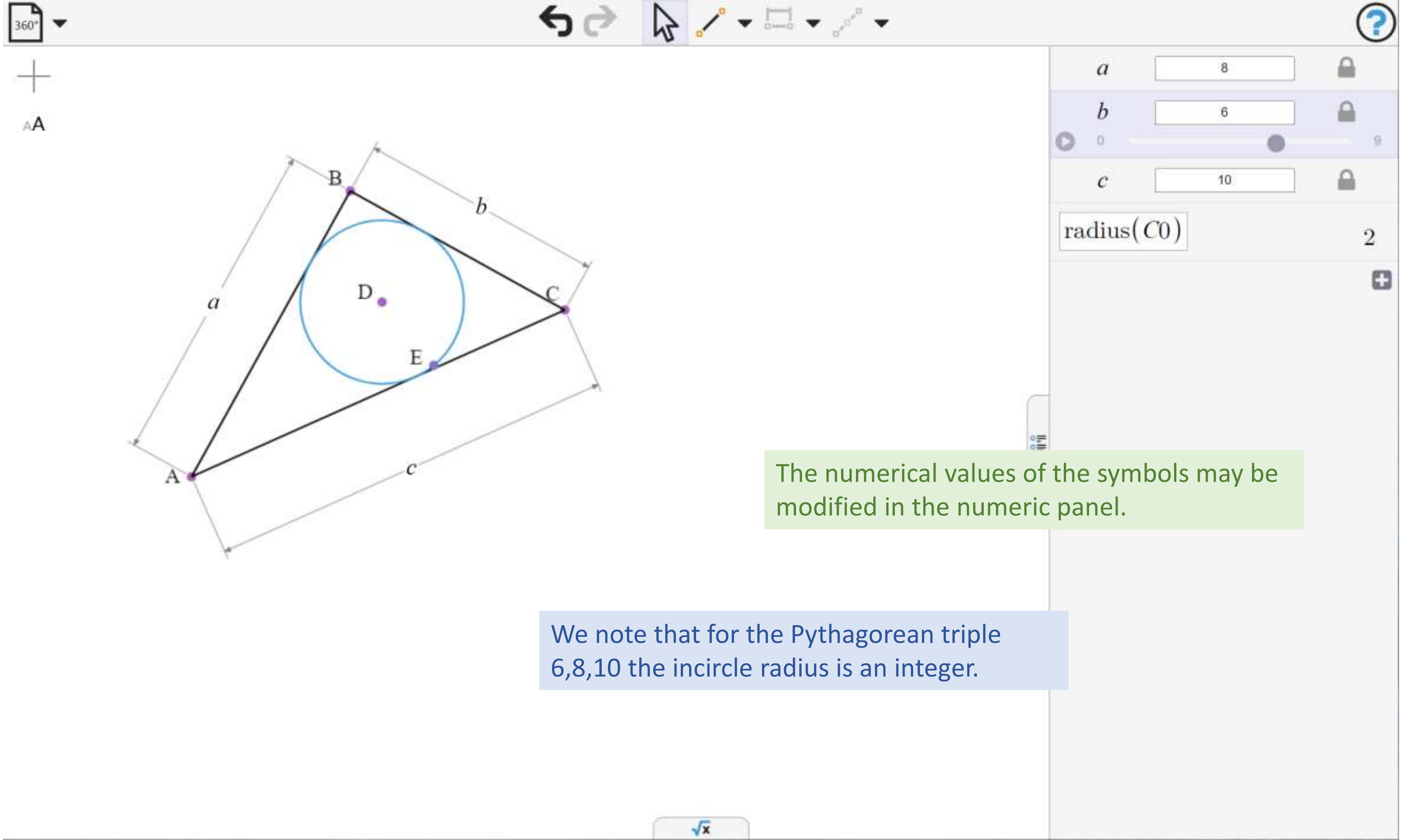

e.g. distance(A,B)
radius(C0)
2.533
B
8
9
D
E
A
C
10
The right hand panel gives numerical measurements: here the radius of the incircle.



The bottom panel gives the measurement symbolically.

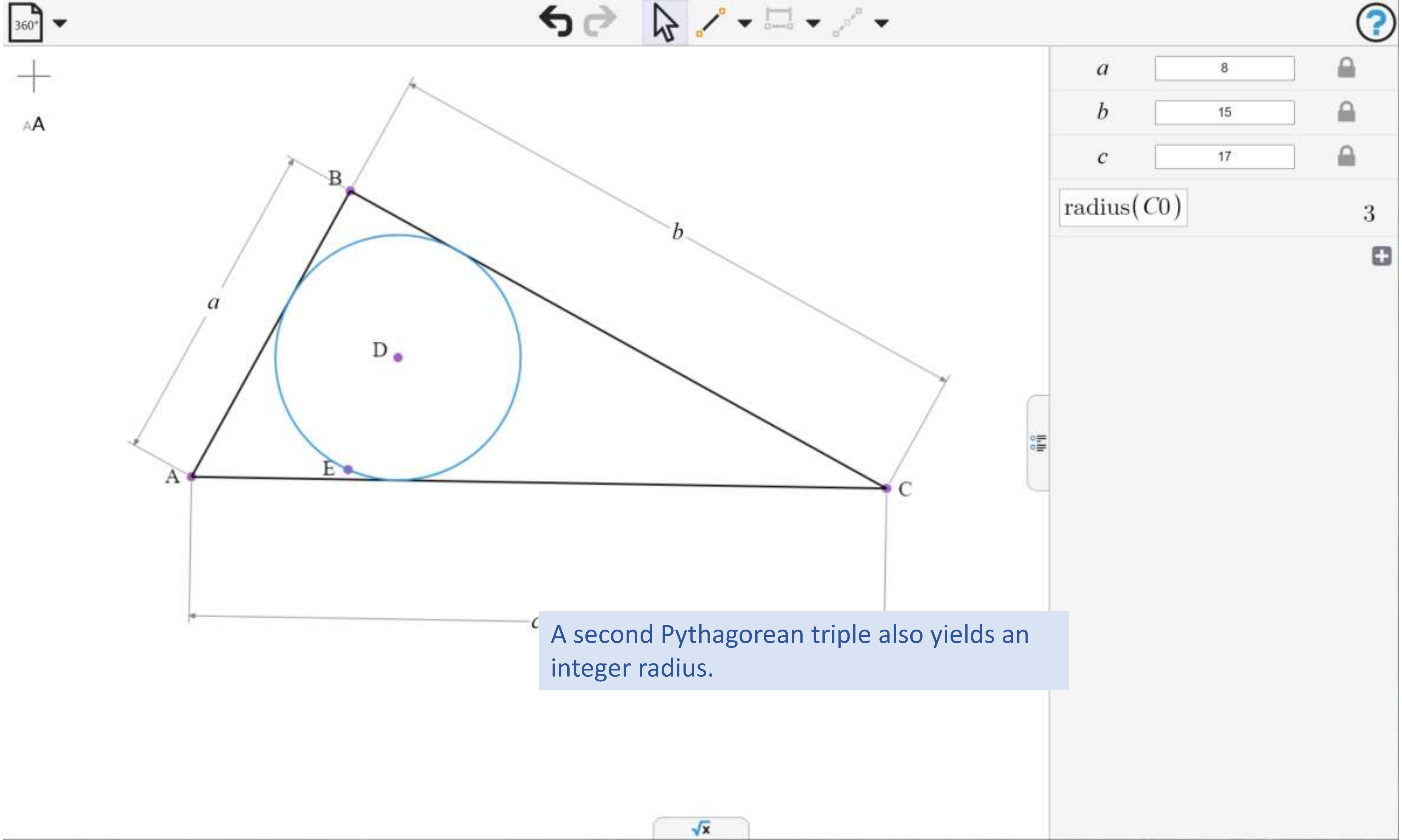


Constraints may be given symbolically. Symbols are given numerical values derived from the drawing.

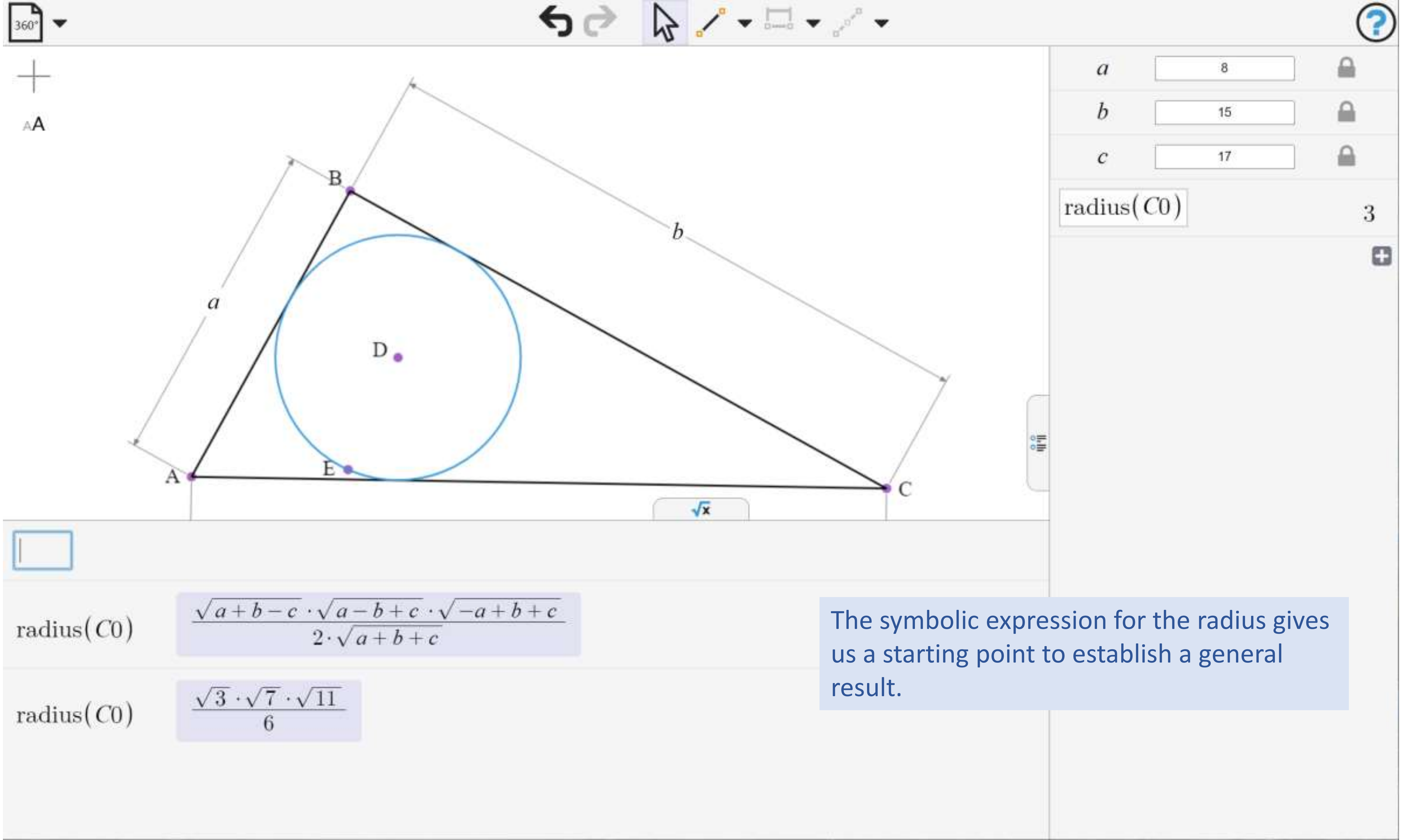


The numerical values of the symbols may be modified in the numeric panel.

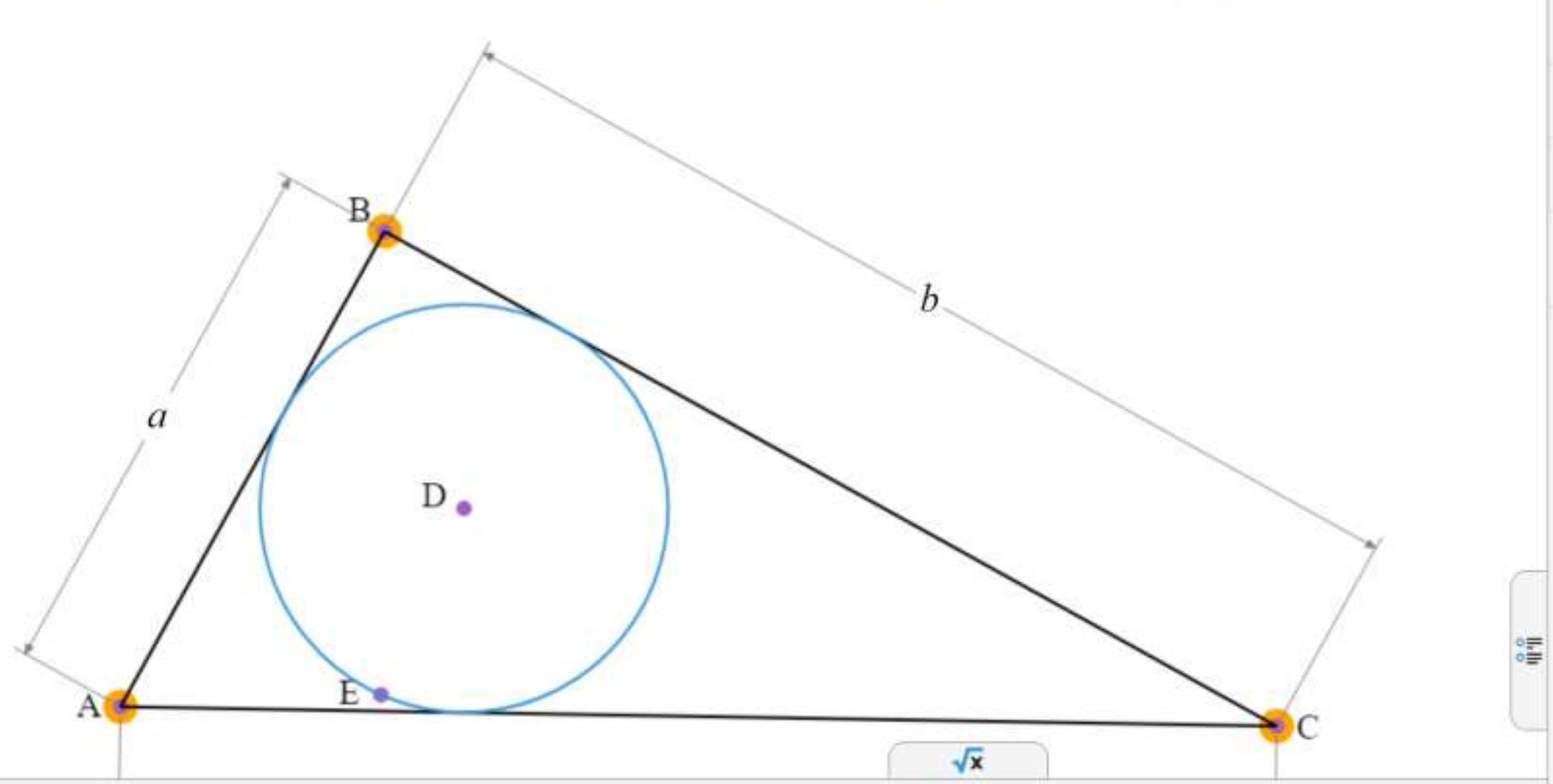
We note that for the Pythagorean triple 6,8,10 the incircle radius is an integer.



A second Pythagorean triple also yields an integer radius.



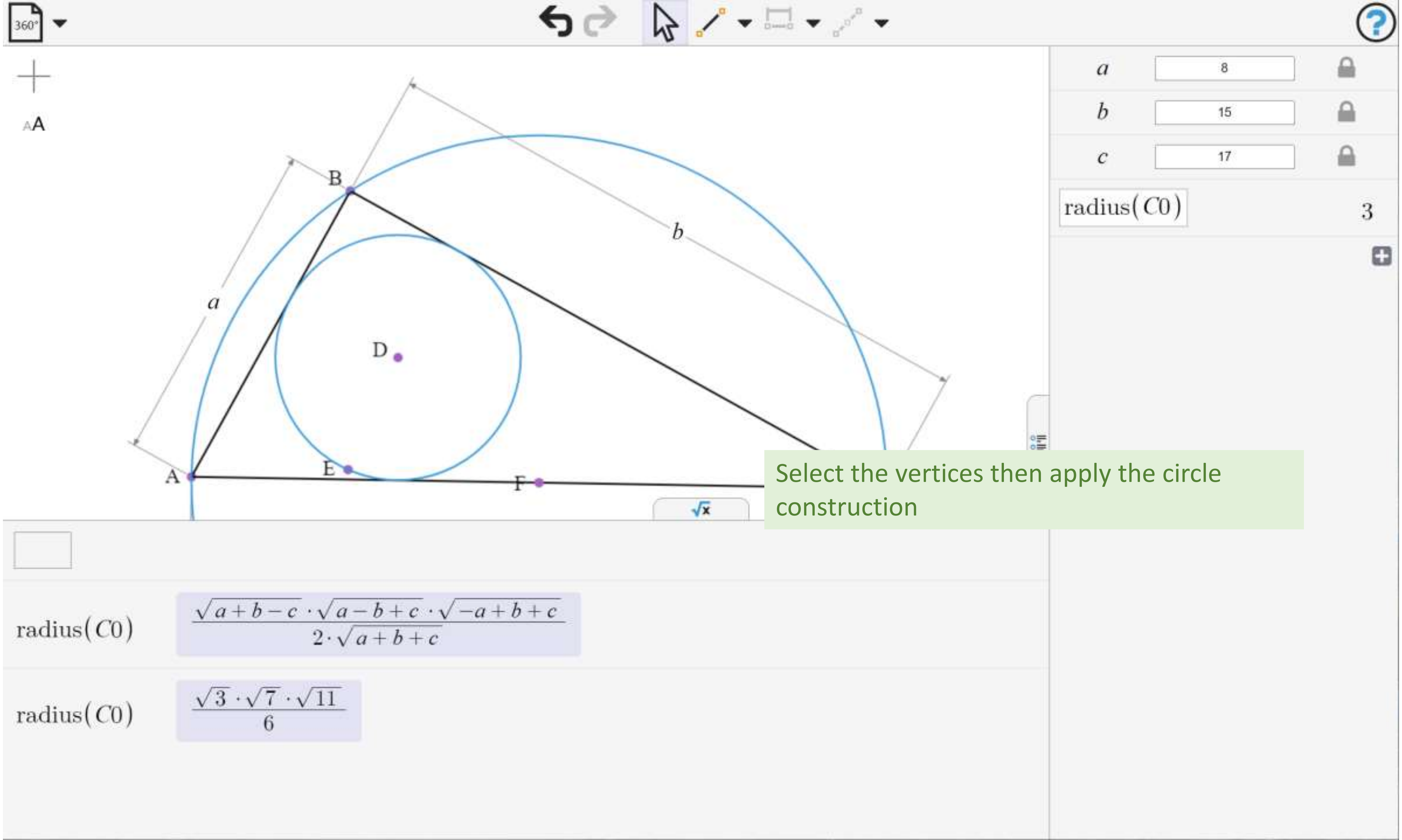
The symbolic expression for the radius gives us a starting point to establish a general result.



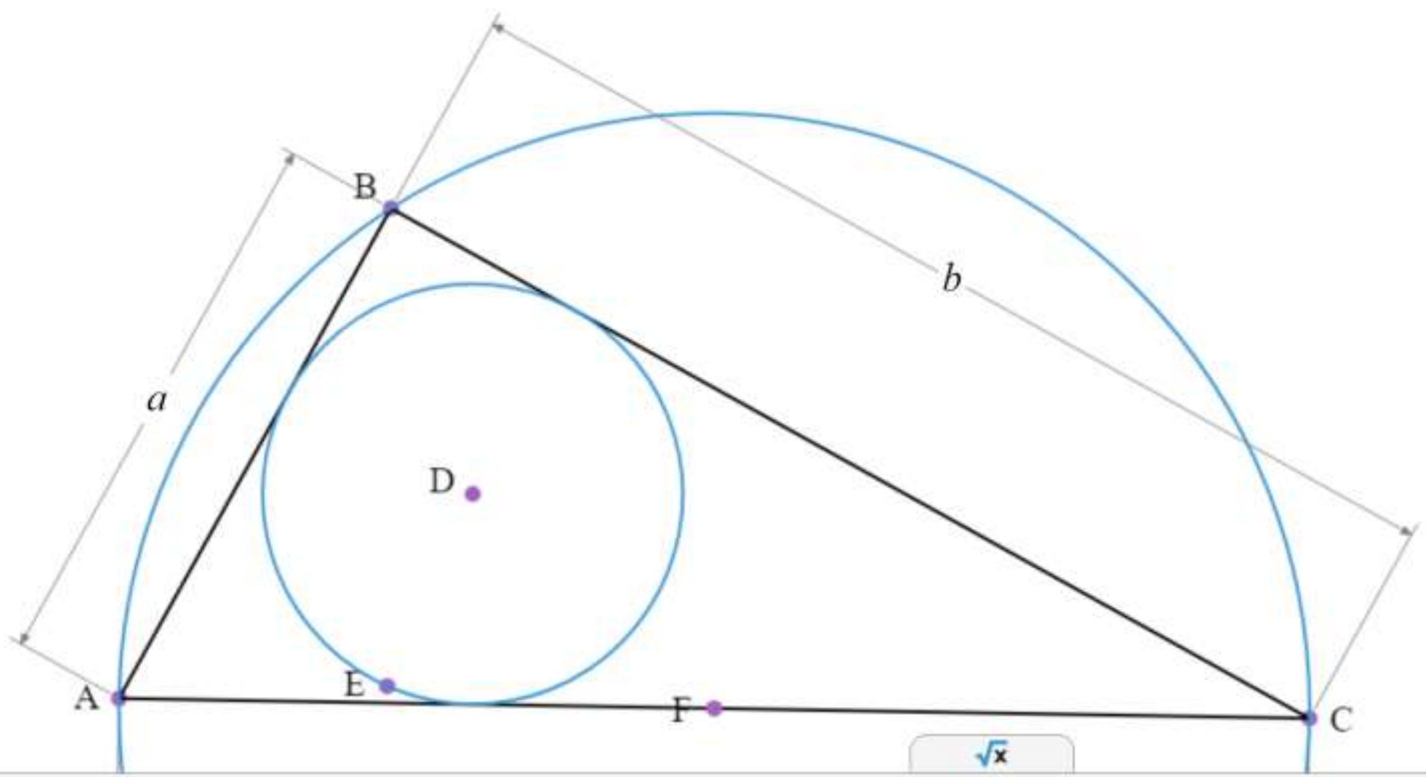
radius(C0) $\frac{\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}}{2\cdot\sqrt{a+b+c}}$

radius(C0) $\frac{\sqrt{3}\cdot\sqrt{7}\cdot\sqrt{11}}{6}$

Meanwhile, let's look at the radius of the circumcircle …



Select the vertices then apply the circle construction



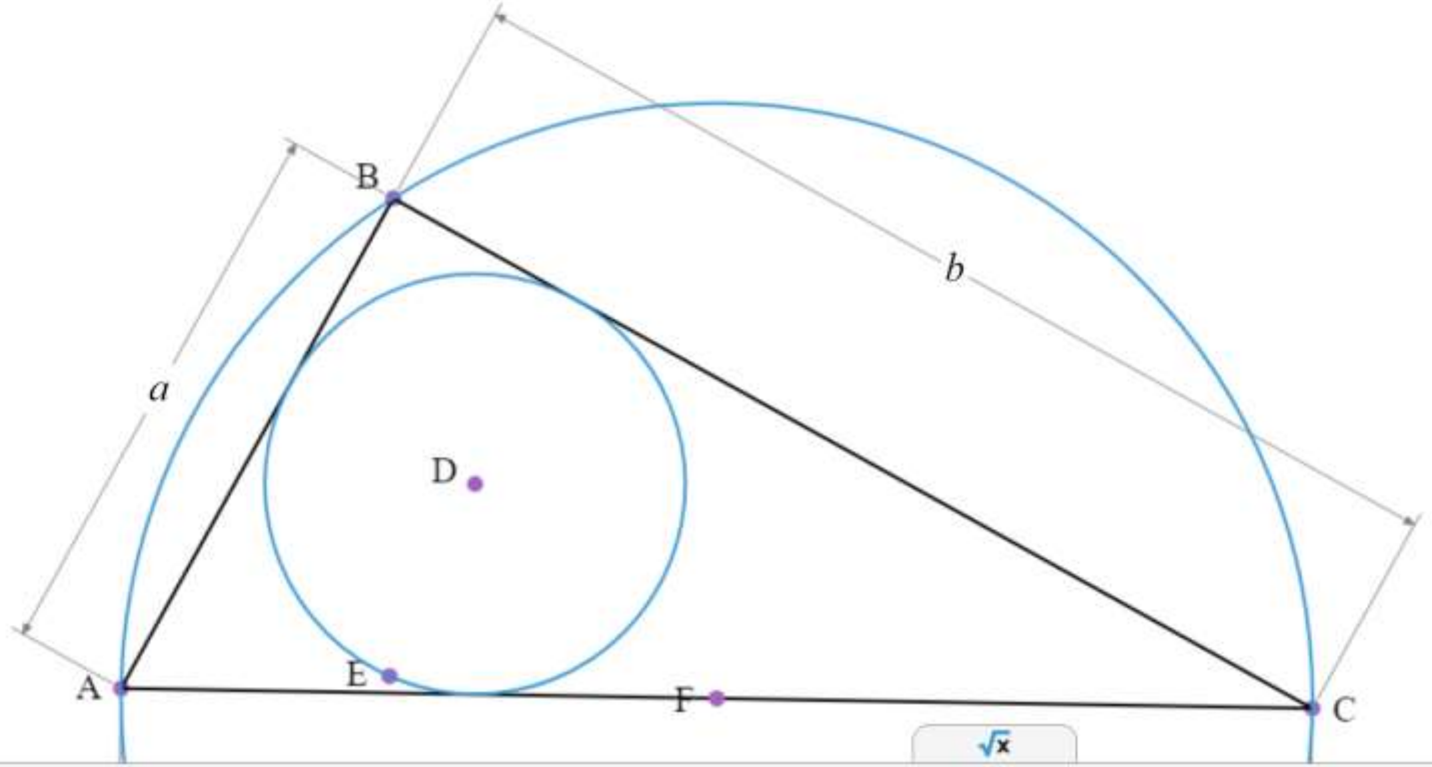
a	8
b	15
c	17
radius($C0$)	3

radius($C1$) $\dfrac{a \cdot b \cdot c}{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}$

radius($C0$) $\dfrac{\sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{2 \cdot \sqrt{a+b+c}}$

radius($C0$) $\dfrac{\sqrt{3} \cdot \sqrt{7} \cdot \sqrt{11}}{6}$

Three terms which are in the numerator of the incircle radius are in the denominator of the circumcircle radius



a = 8

b = 15

c = 17

radius(C0) 3

$$\text{radius}(C0)\cdot\text{radius}(C1) \quad \frac{a\cdot b\cdot c}{2\cdot(a+b+c)}$$

$$\text{radius}(C1) \quad \frac{a\cdot b\cdot c}{\sqrt{a+b+c}\cdot\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}}$$

$$\text{radius}(C0) \quad \frac{\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}}{2\cdot\sqrt{a+b+c}}$$

Which suggests that the product of the radii simplifies.

Take-away questions

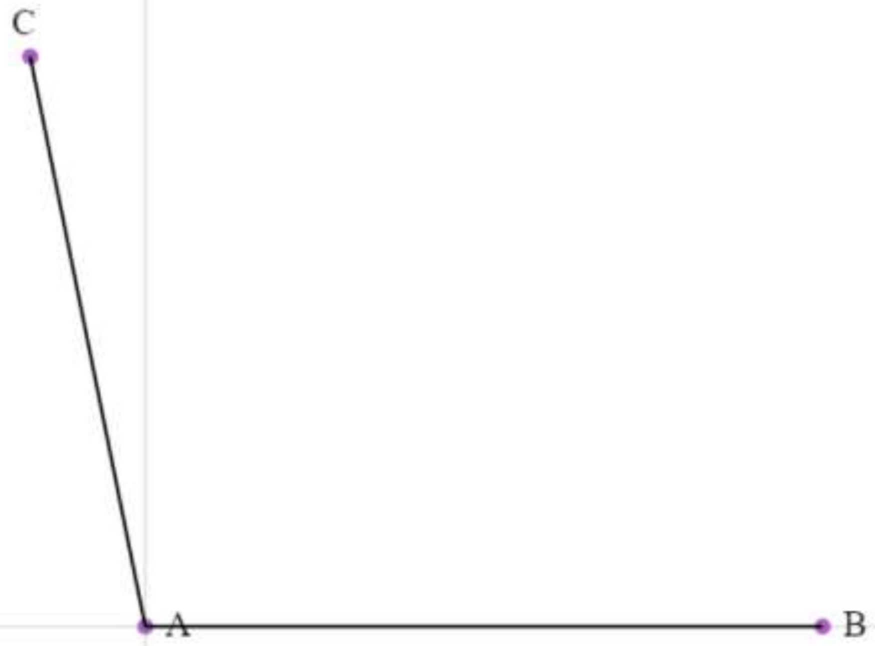
Is the incircle radius of a Pythagorean triangle always an integer?

How about the excircles?

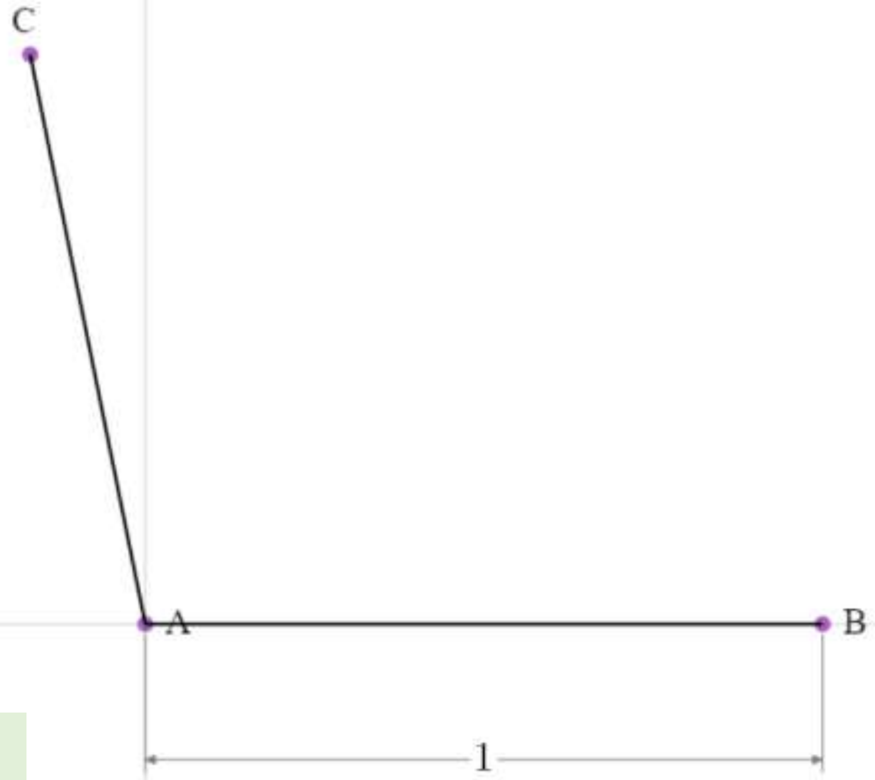
Can we discover any formulas connecting the incircle and excircle radii?

Box solar cooker –
mathematical modelling

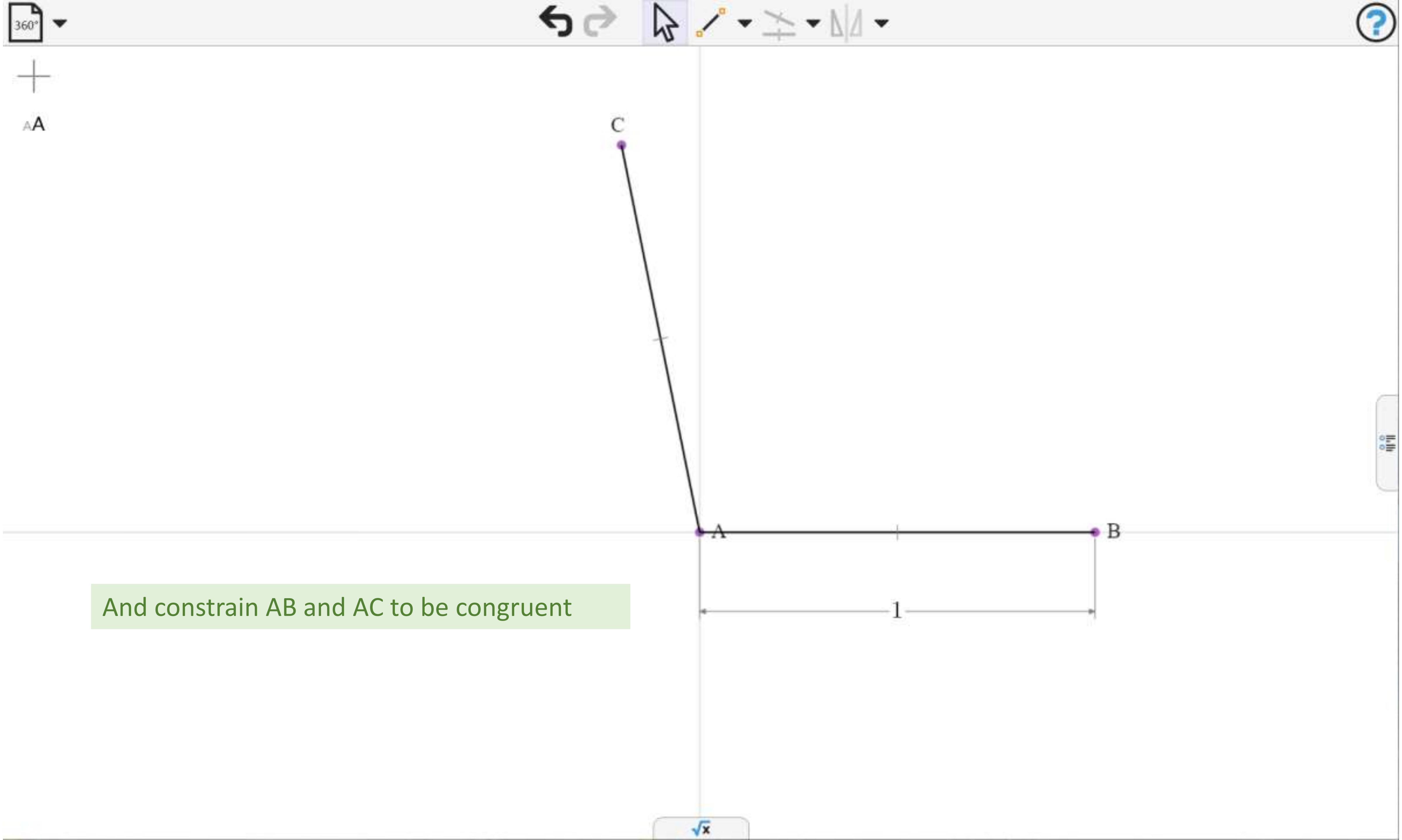
Algebra 1



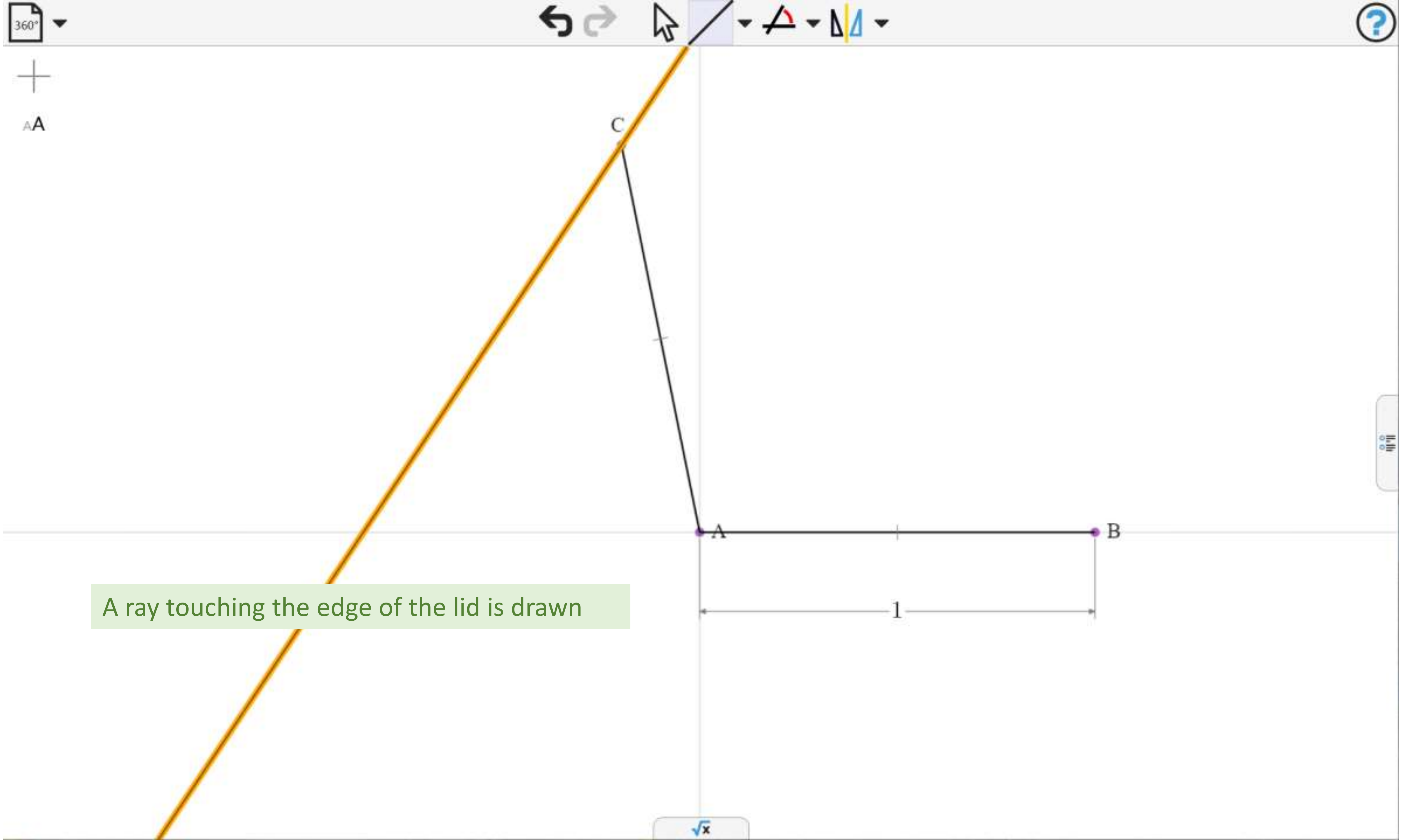
Sketch the box top and lid, putting the hinge at the origin, and the box top along the x axis.



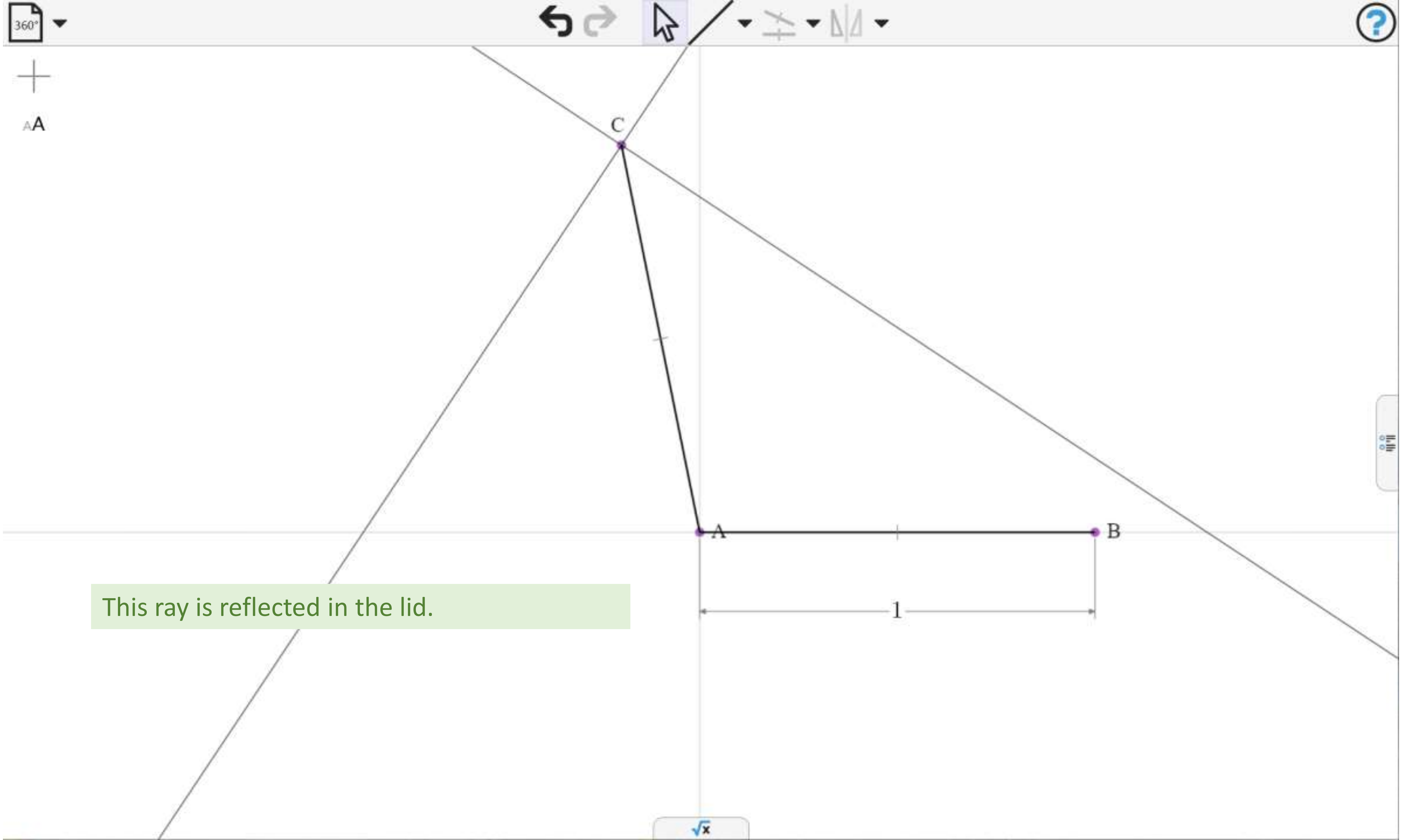
We’ll make the box length 1.



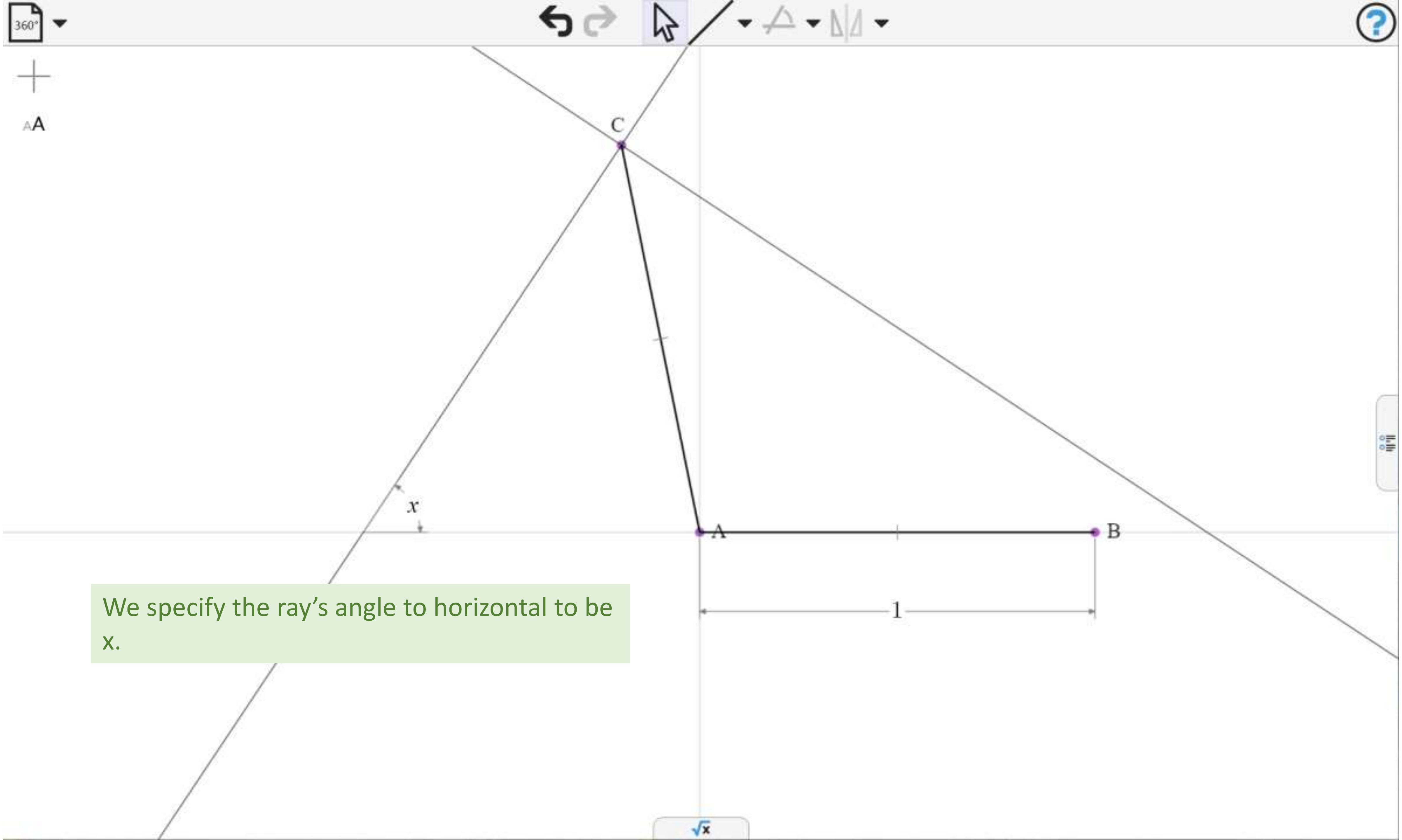
And constrain AB and AC to be congruent



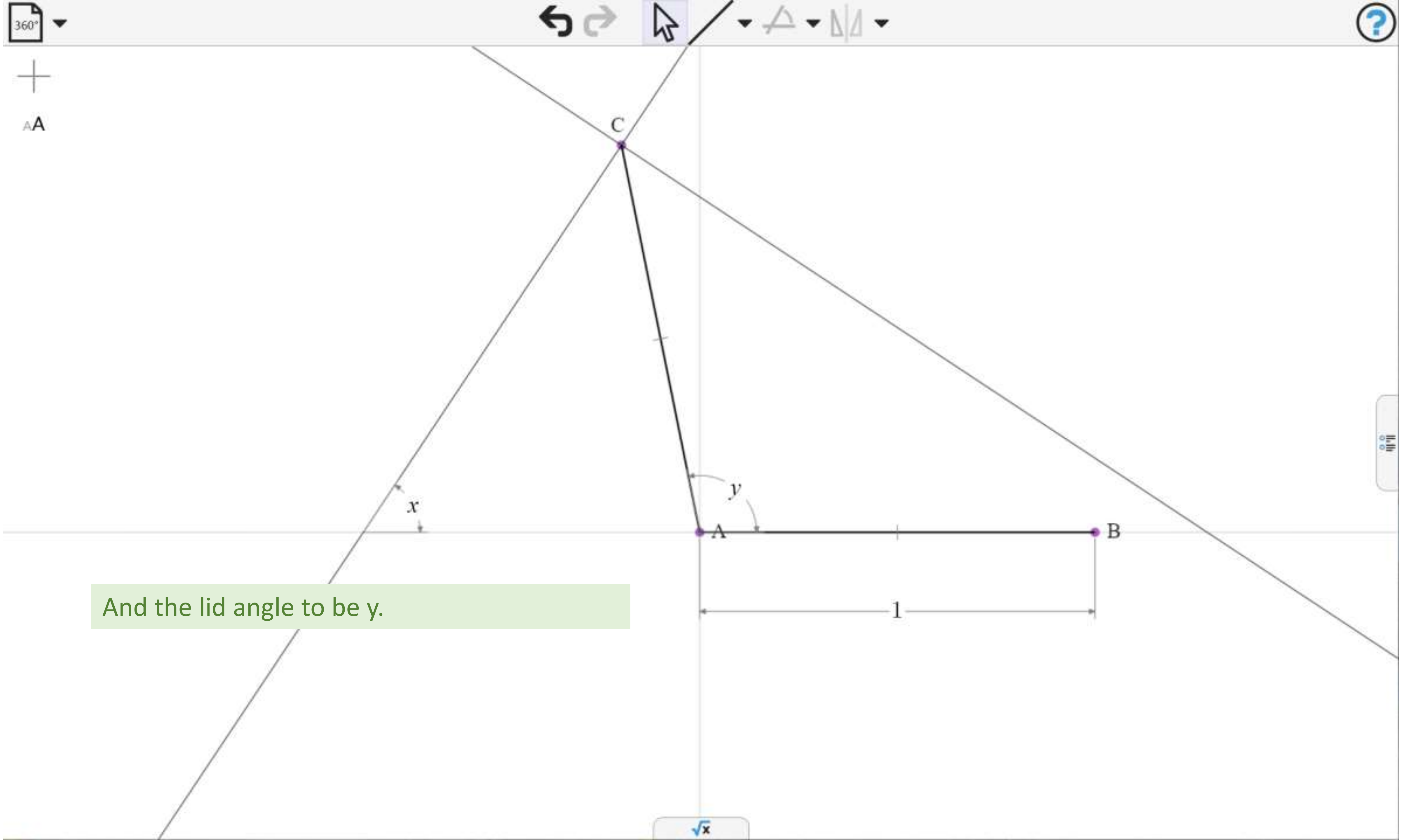
A ray touching the edge of the lid is drawn



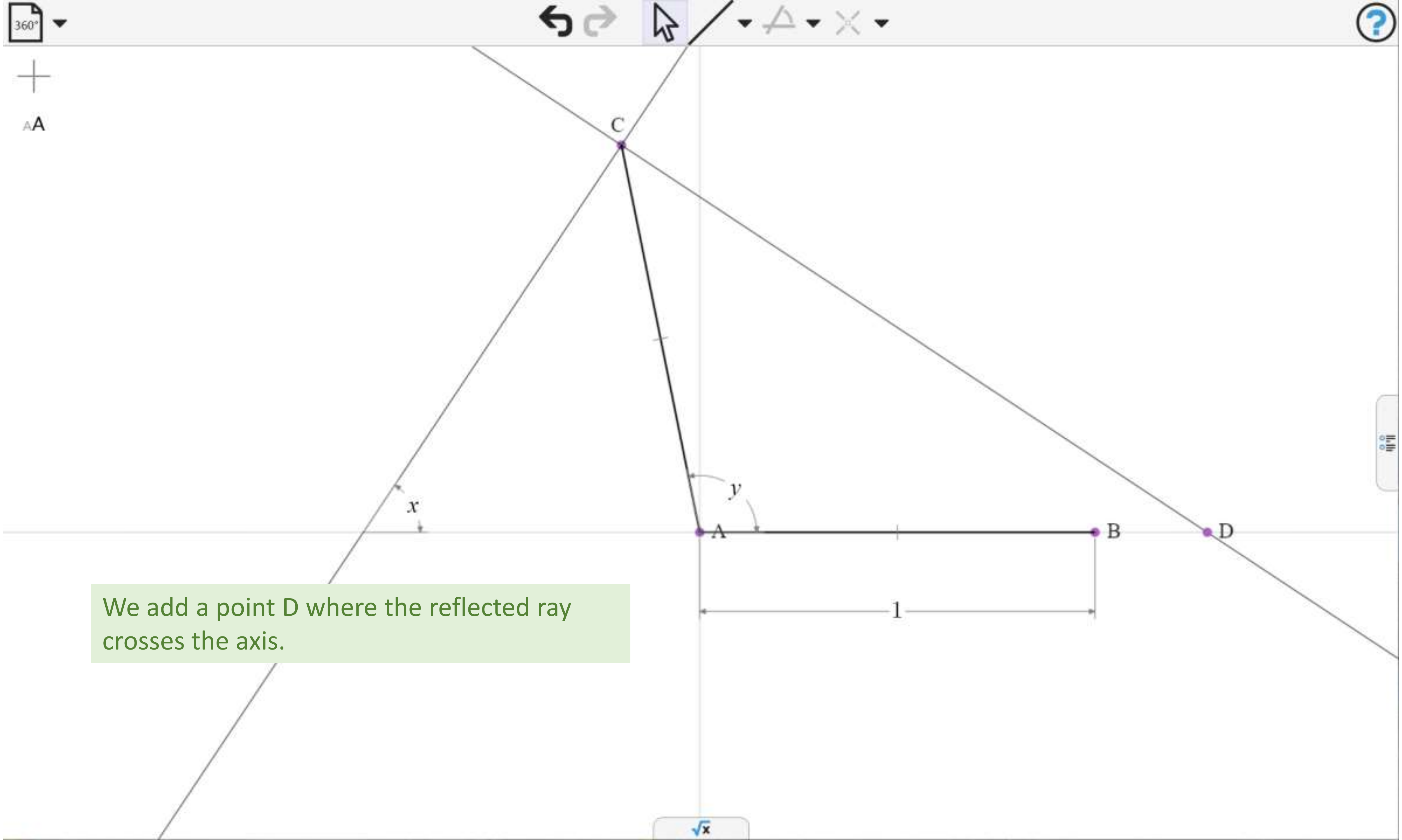
This ray is reflected in the lid.



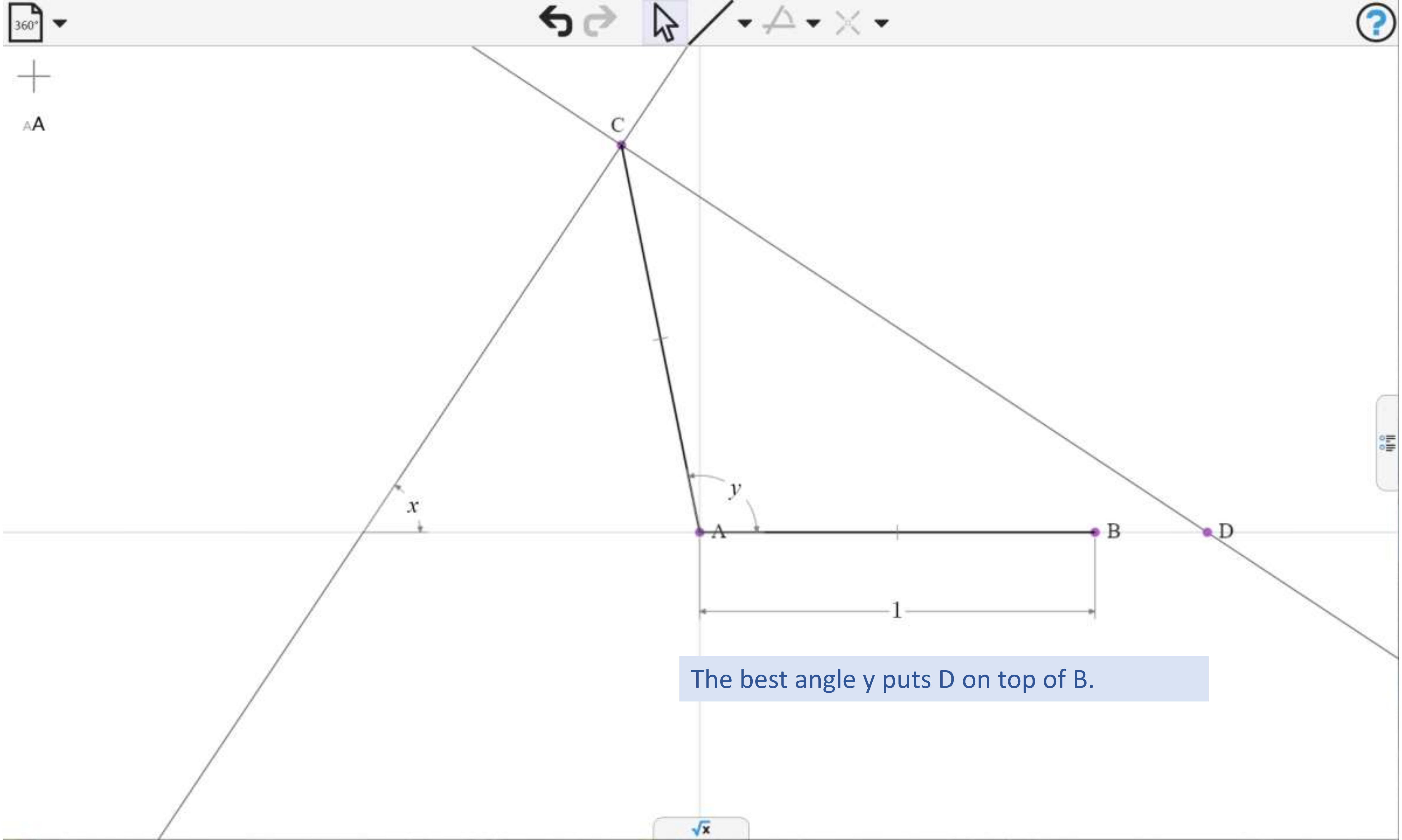
We specify the ray’s angle to horizontal to be x.



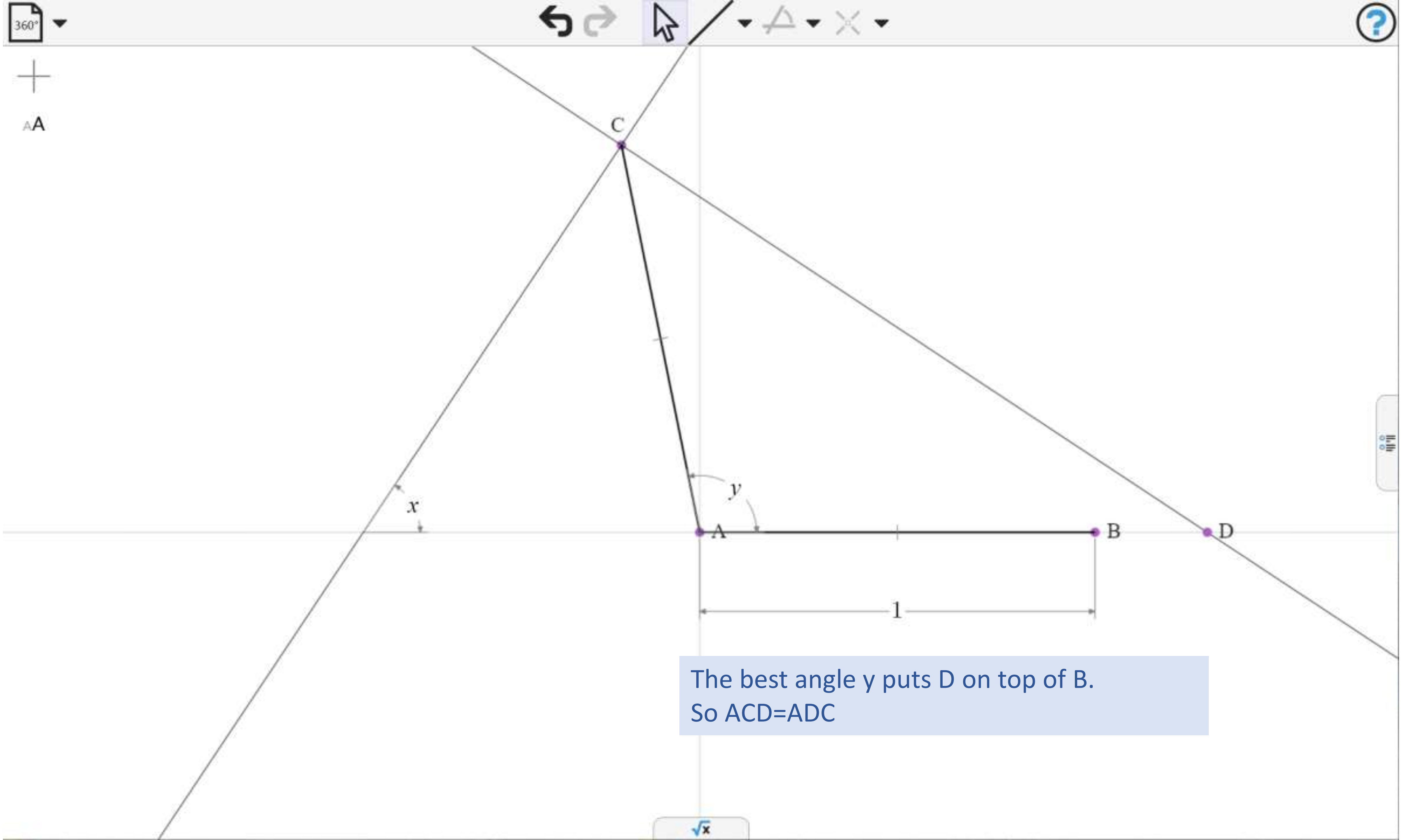
And the lid angle to be y.



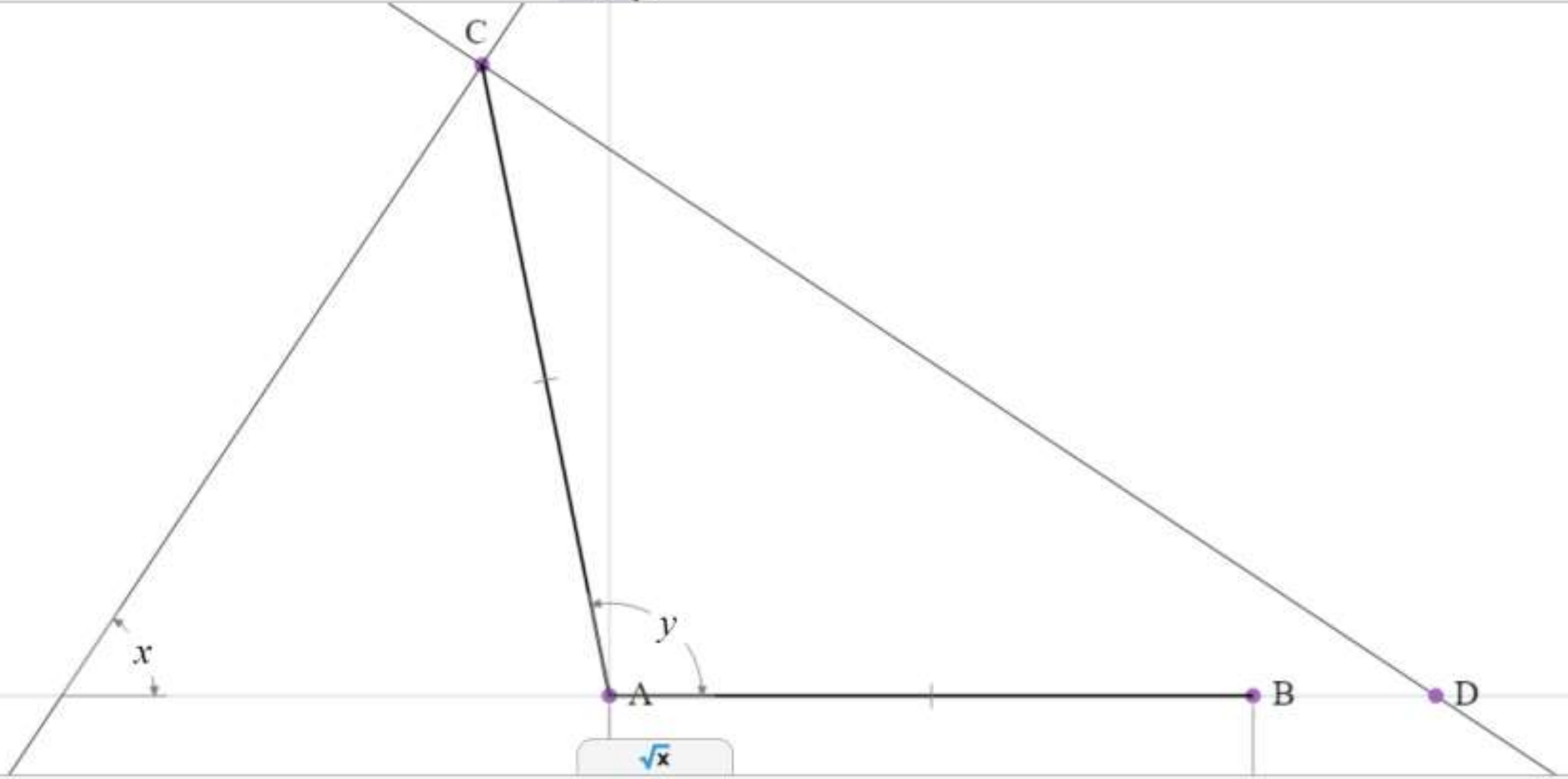
We add a point D where the reflected ray crosses the axis.



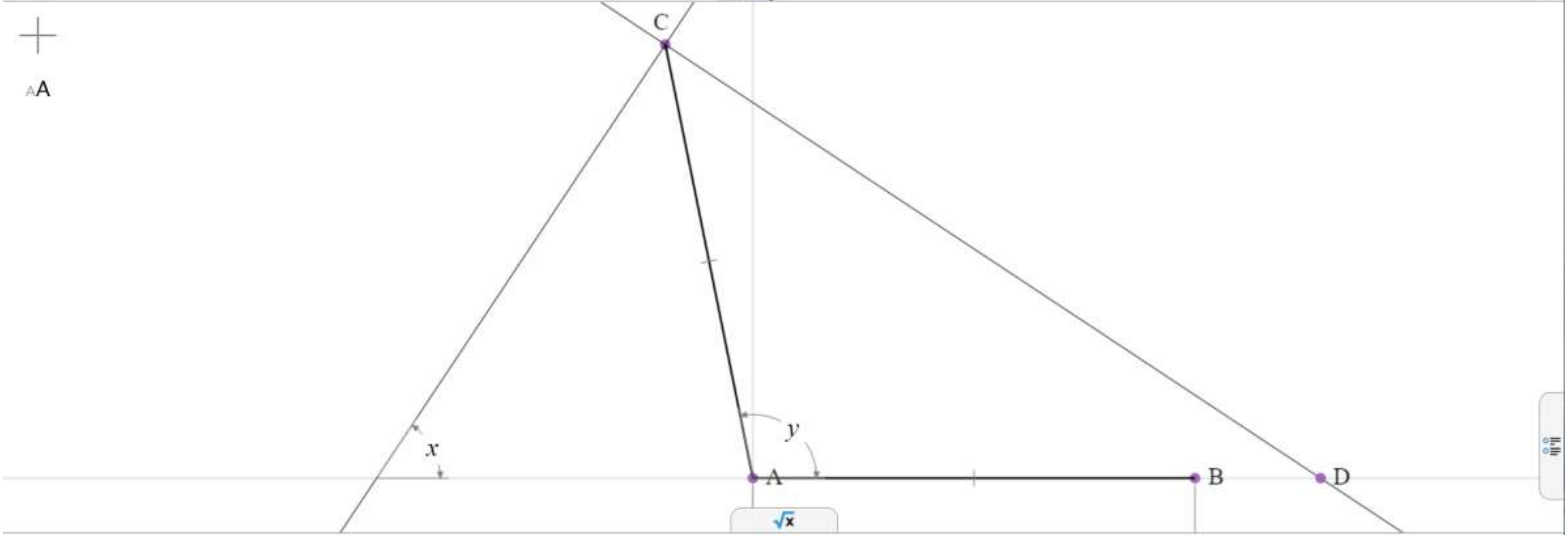
The best angle y puts D on top of B.



The best angle y puts D on top of B.
So ACD=ADC



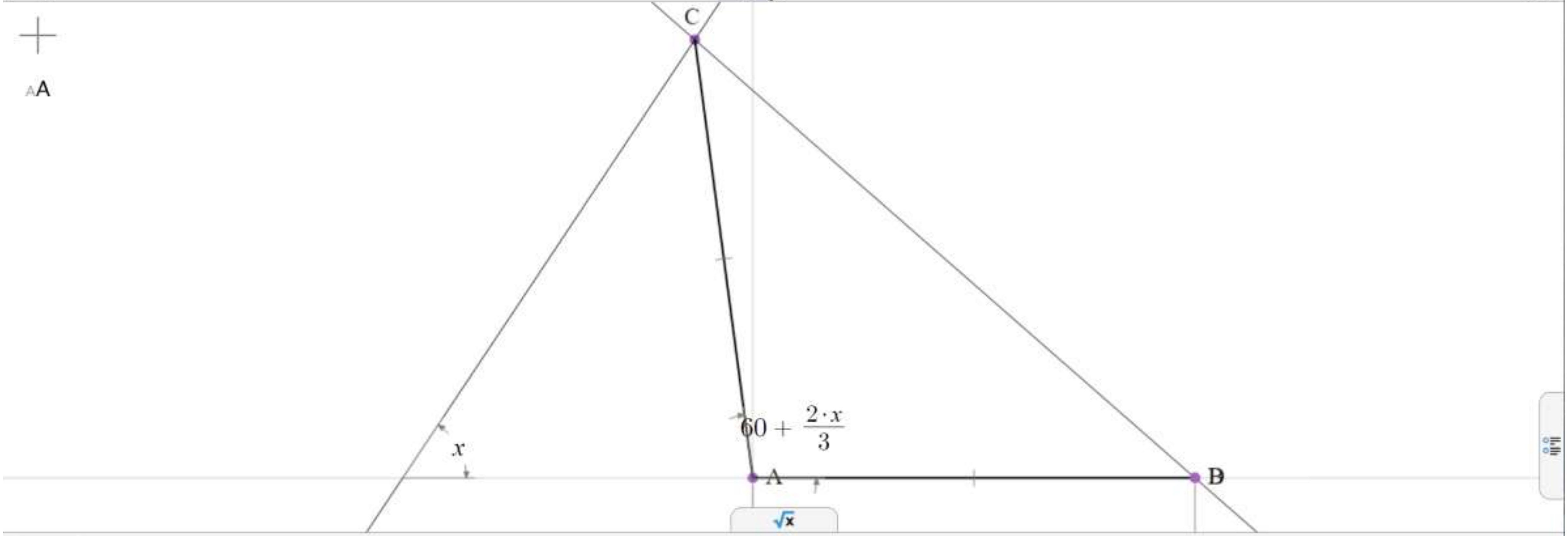
angle(A,C,D) $-x+y$



angle(A,D,C) $180 + x - 2 \cdot y$

angle(A,C,D) $-x + y$

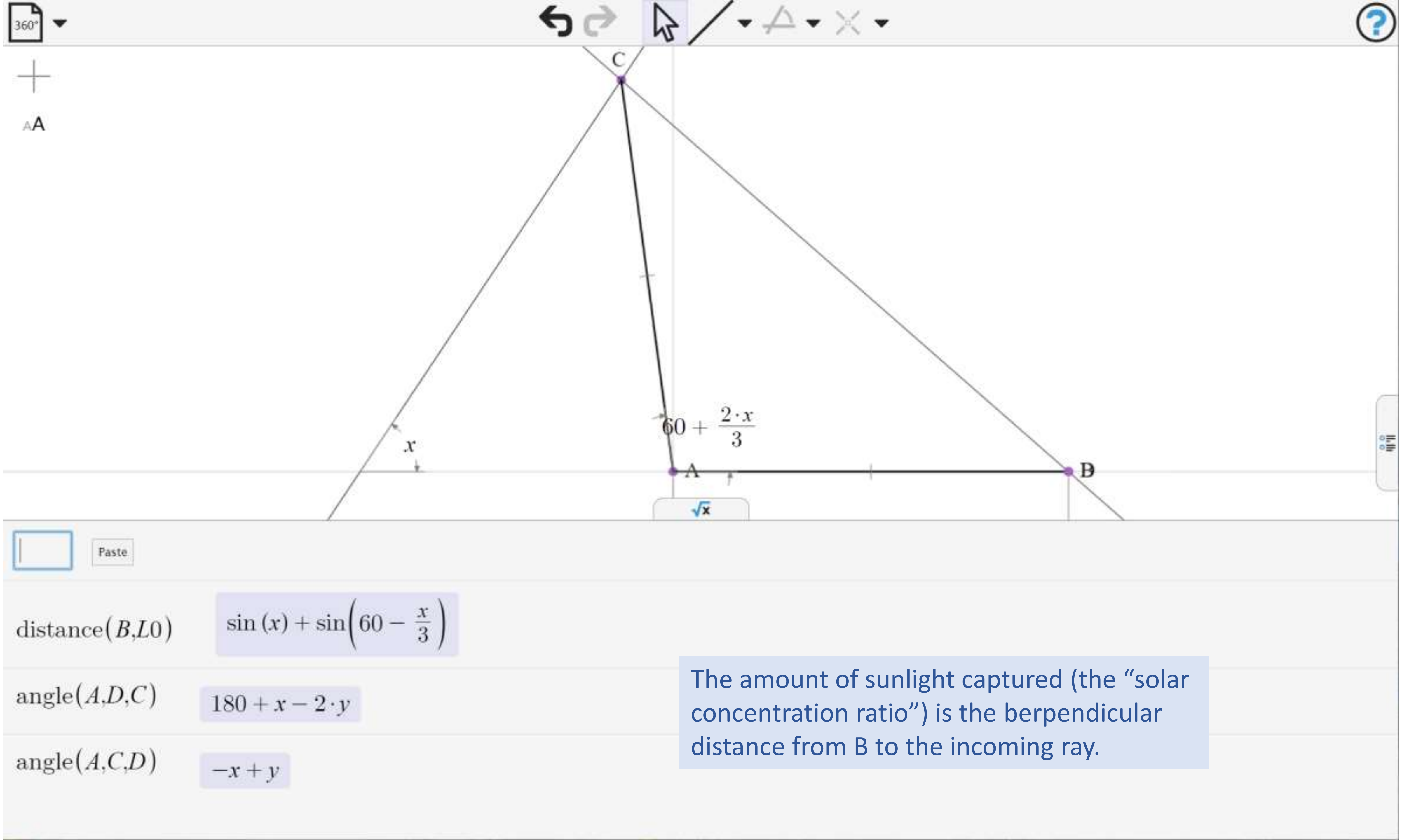
Equate these angles to give a simple equation for y.



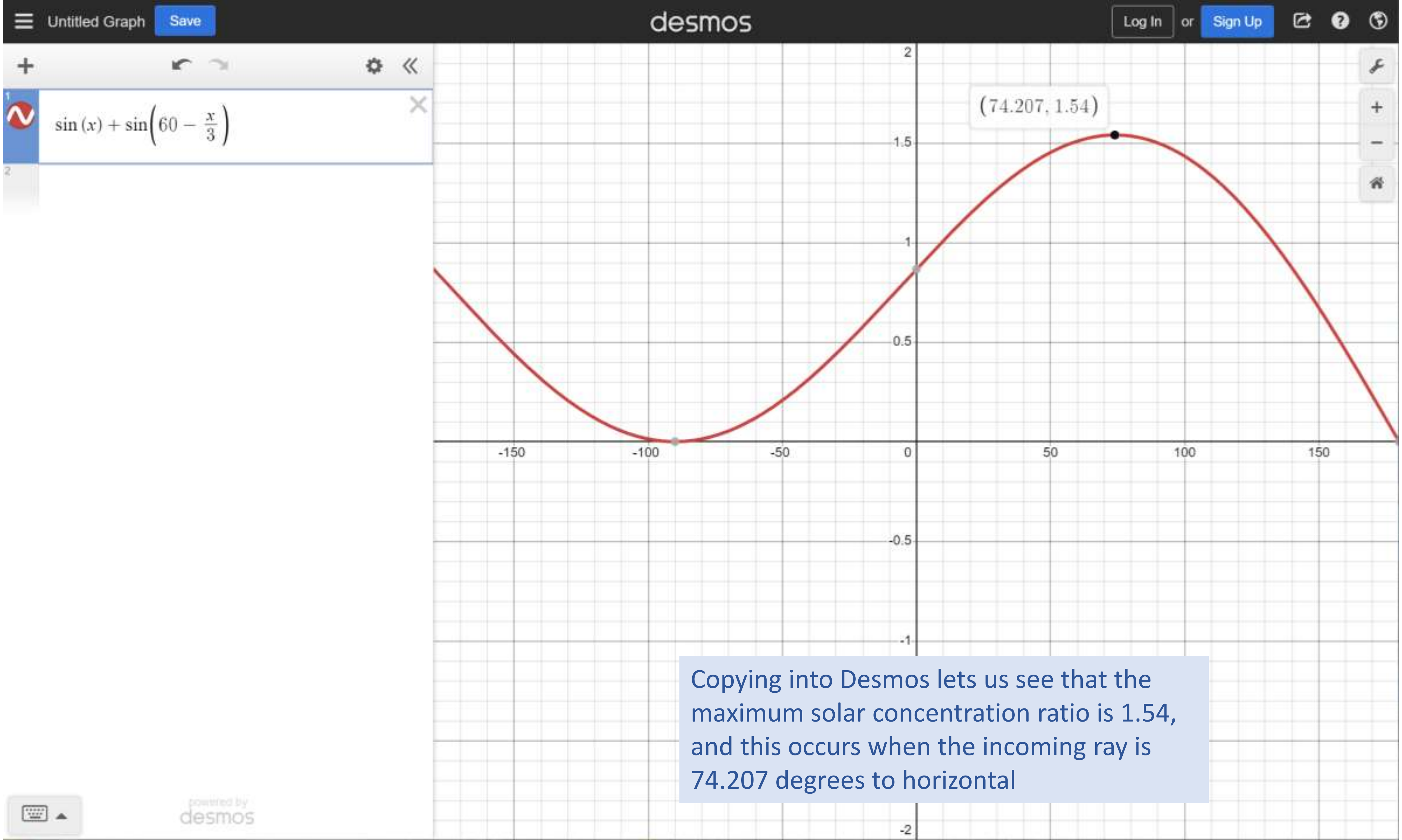
angle(A,D,C) $180 + x - 2 \cdot y$

angle(A,C,D) $-x + y$

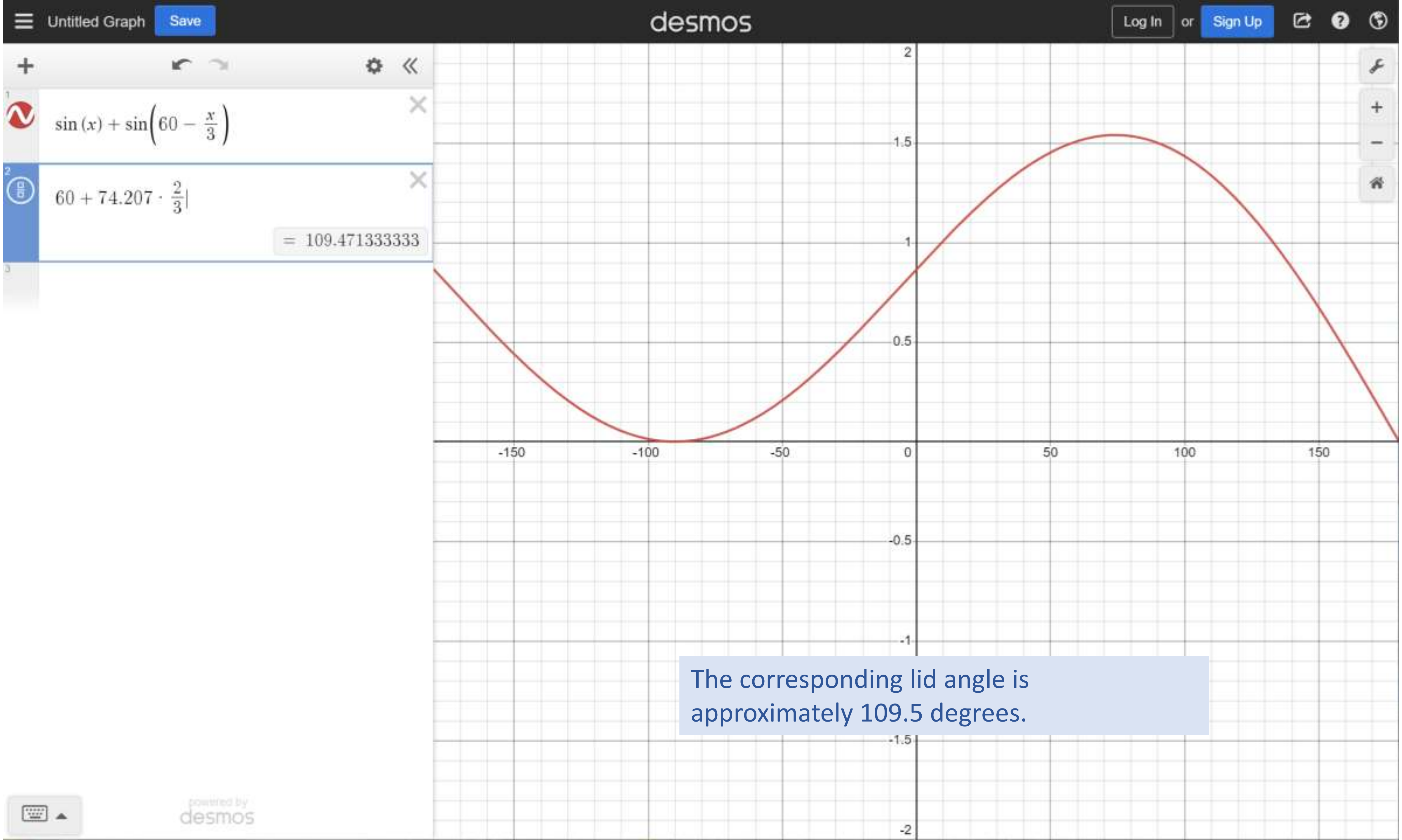
Whose solution is y=60+2x/3



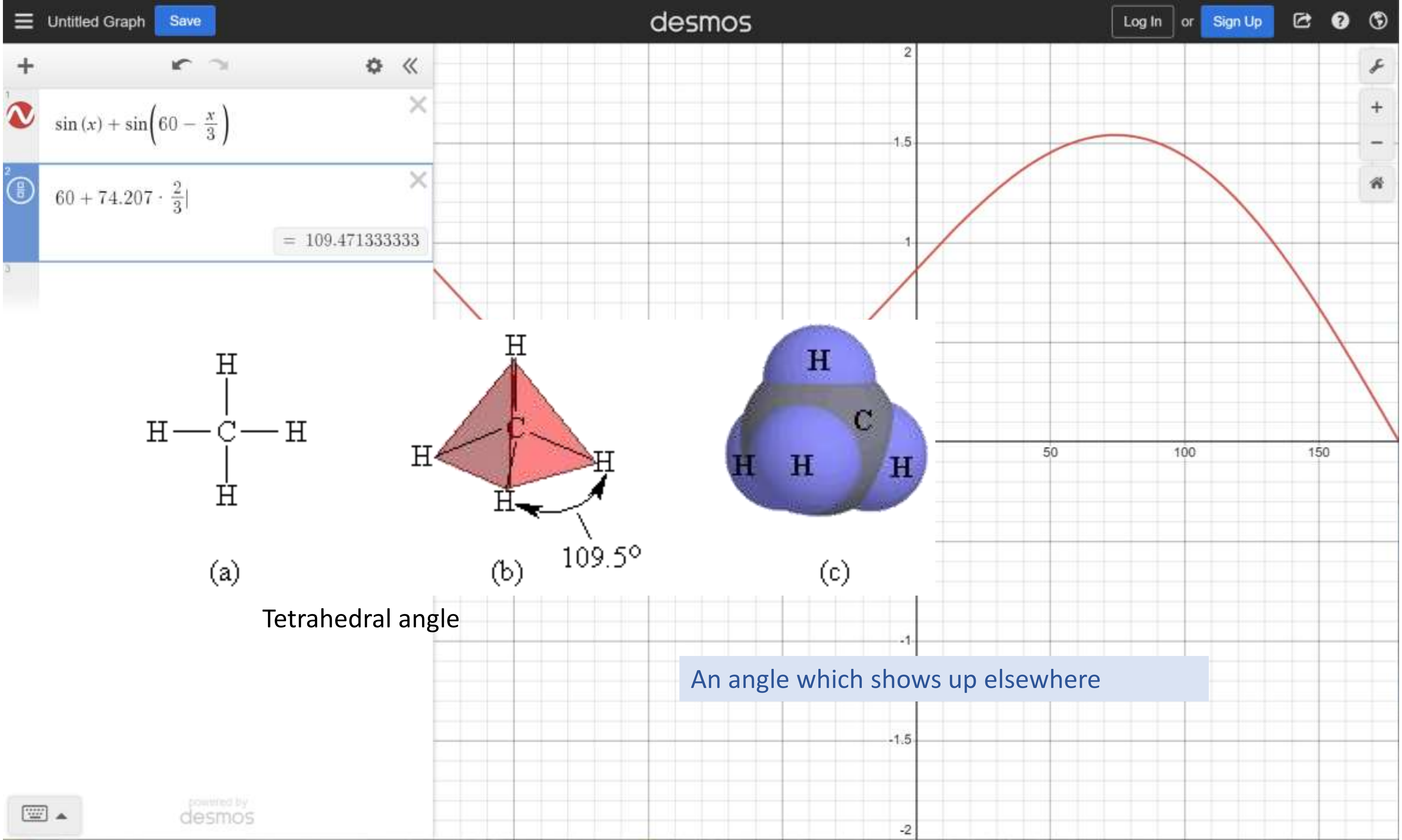
The amount of sunlight captured (the "solar concentration ratio") is the berpendicular distance from B to the incoming ray.



Copying into Desmos lets us see that the maximum solar concentration ratio is 1.54, and this occurs when the incoming ray is 74.207 degrees to horizontal



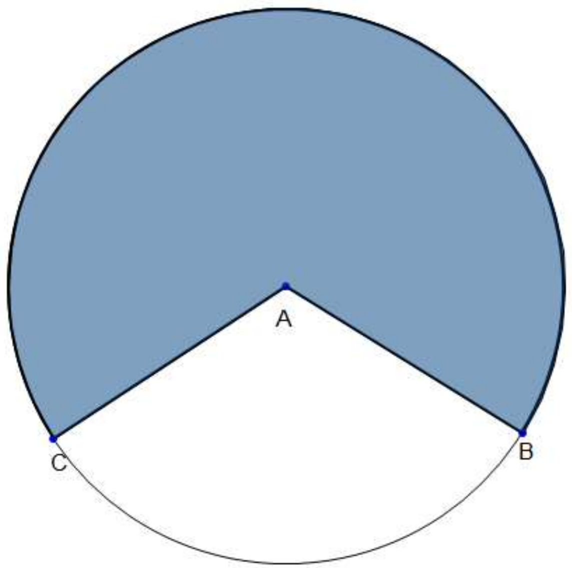
The corresponding lid angle is approximately 109.5 degrees.

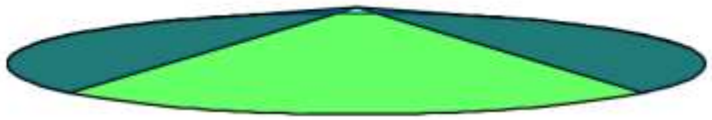


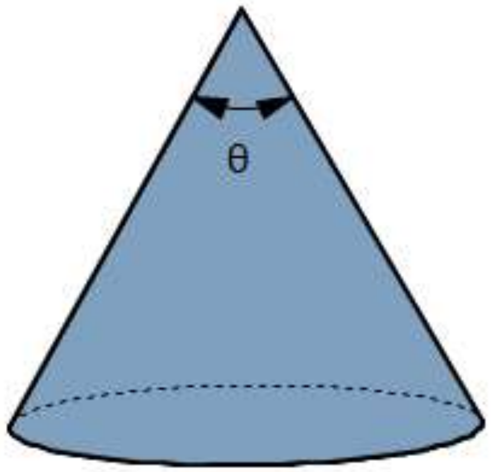
Tetrahedral angle

An angle which shows up elsewhere

Angle at apex of largest paper cone







An angle which shows up elsewhere in more than one place

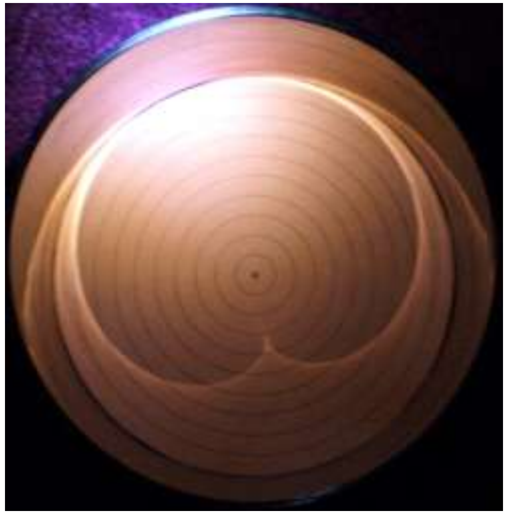
Take-away question

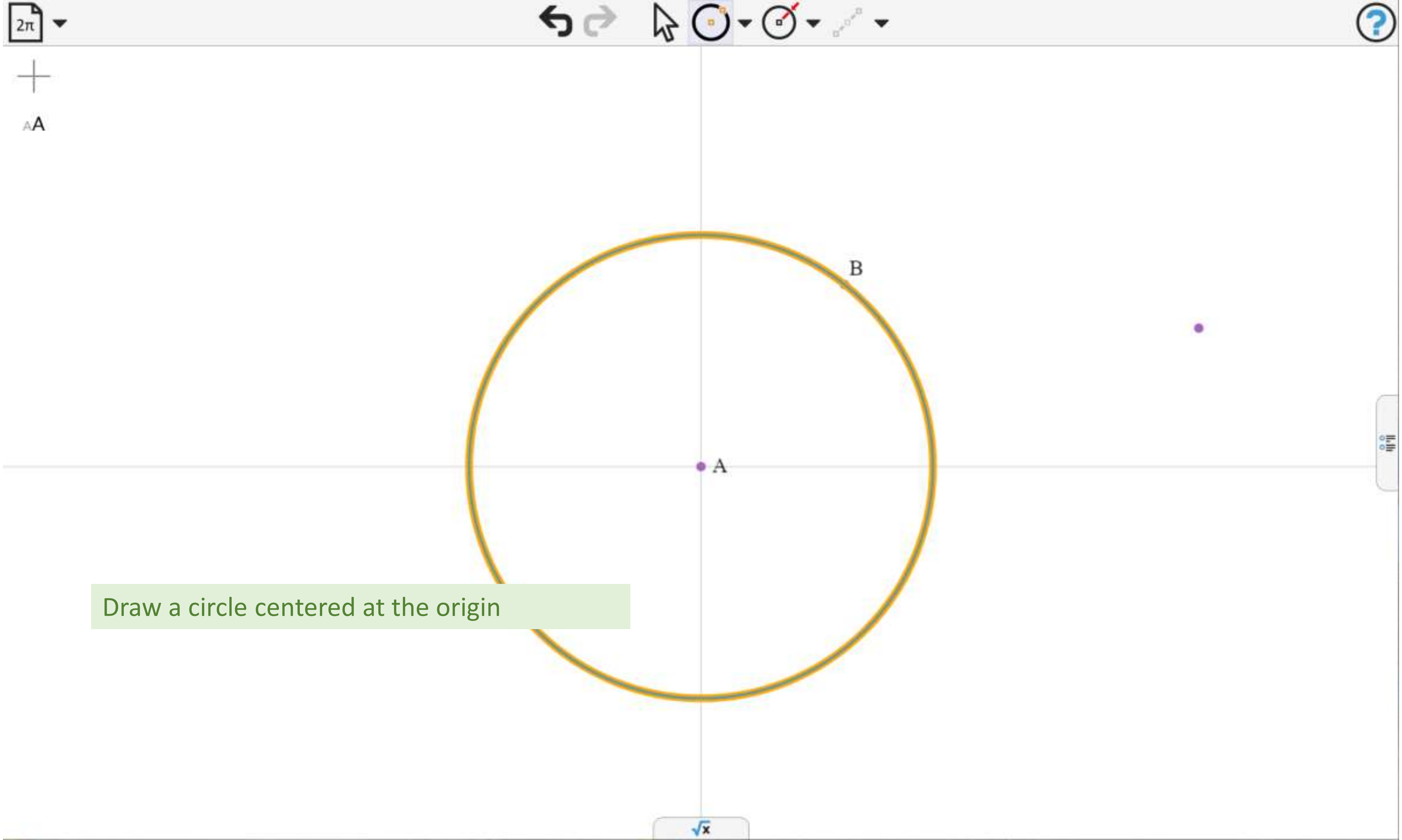
We found numerical values for the most sunlight captured, and the angle at which it is attained.

Can you derive exact values?

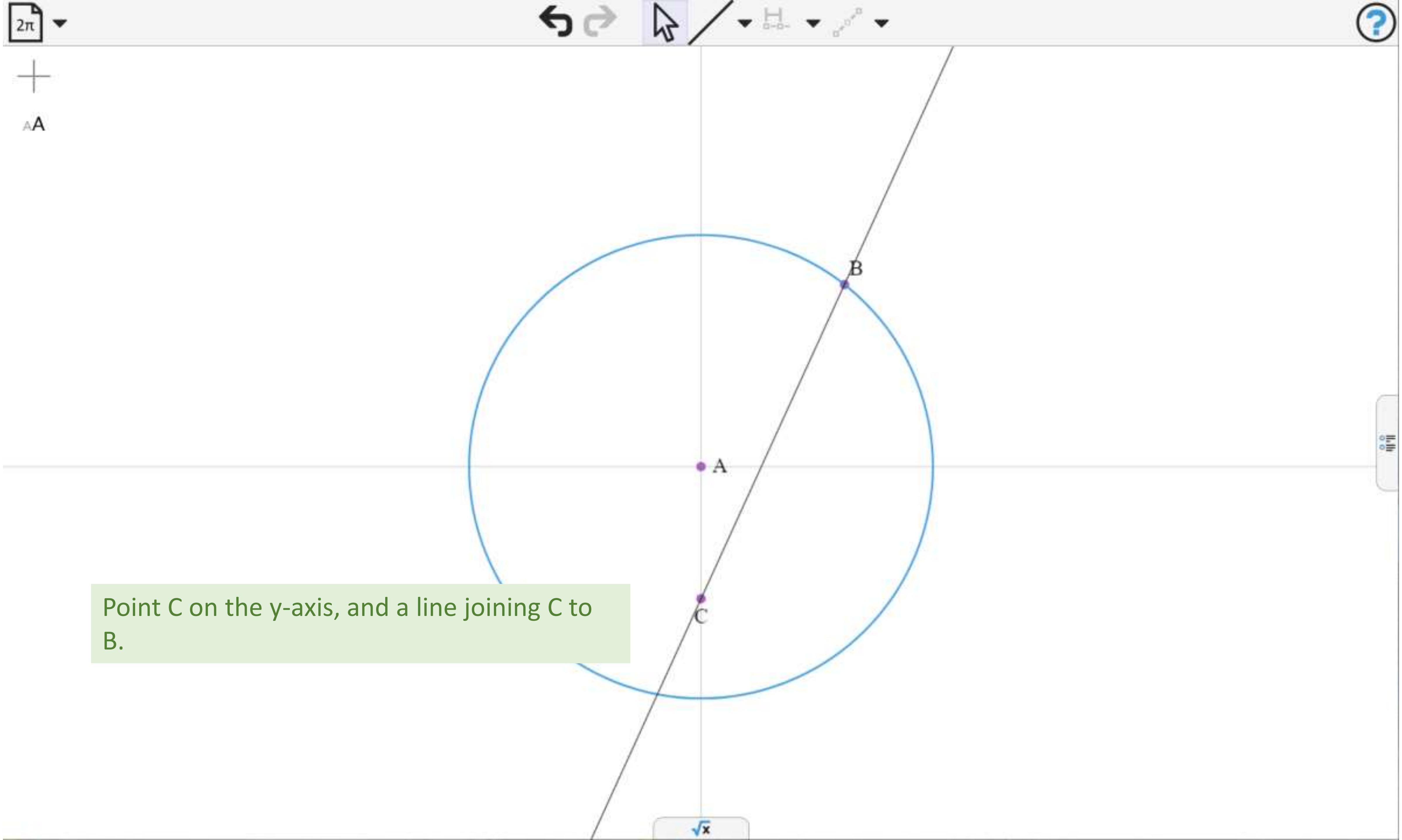
Circle caustics – mathematical modelling
trig / precalculus

Point light source

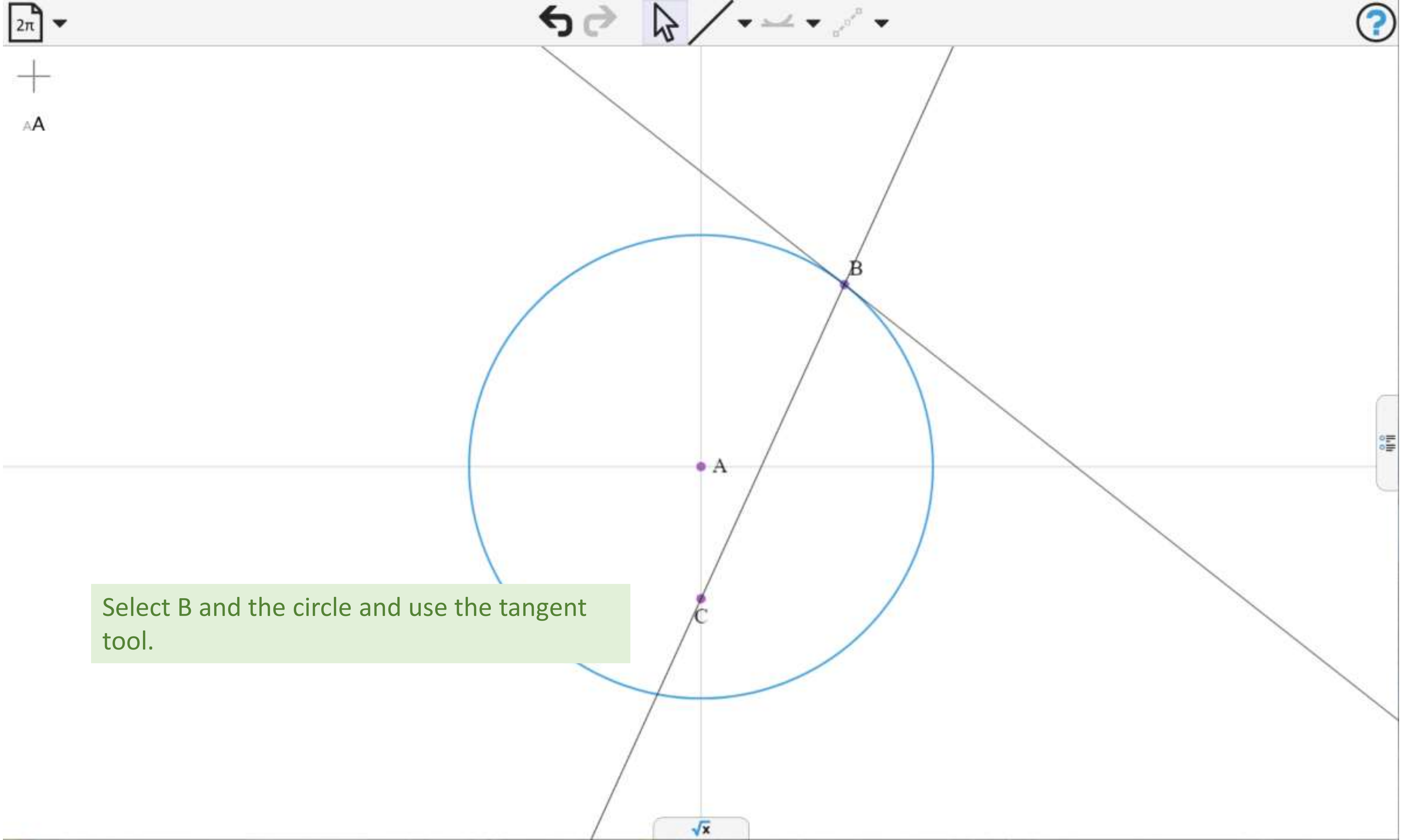




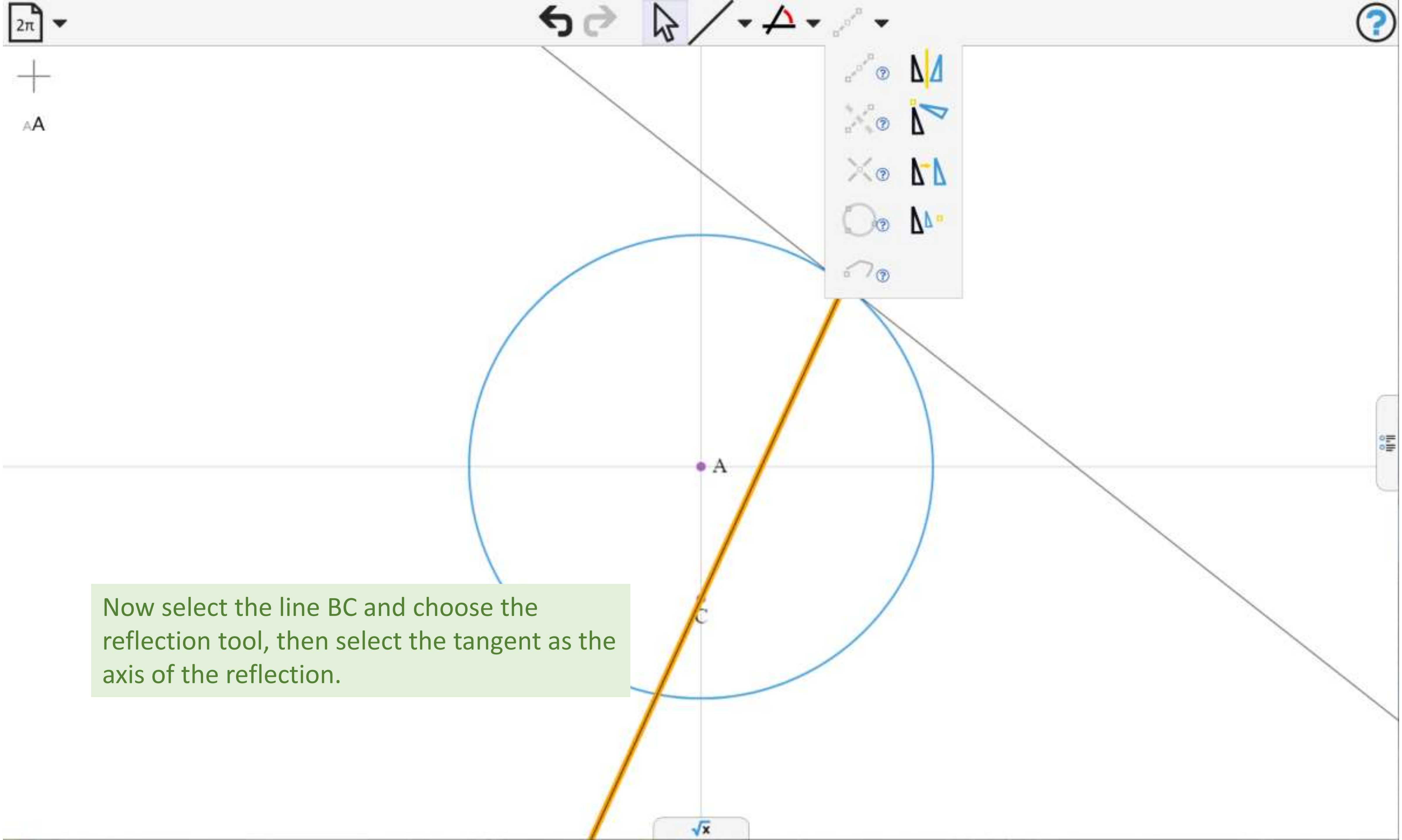
Draw a circle centered at the origin



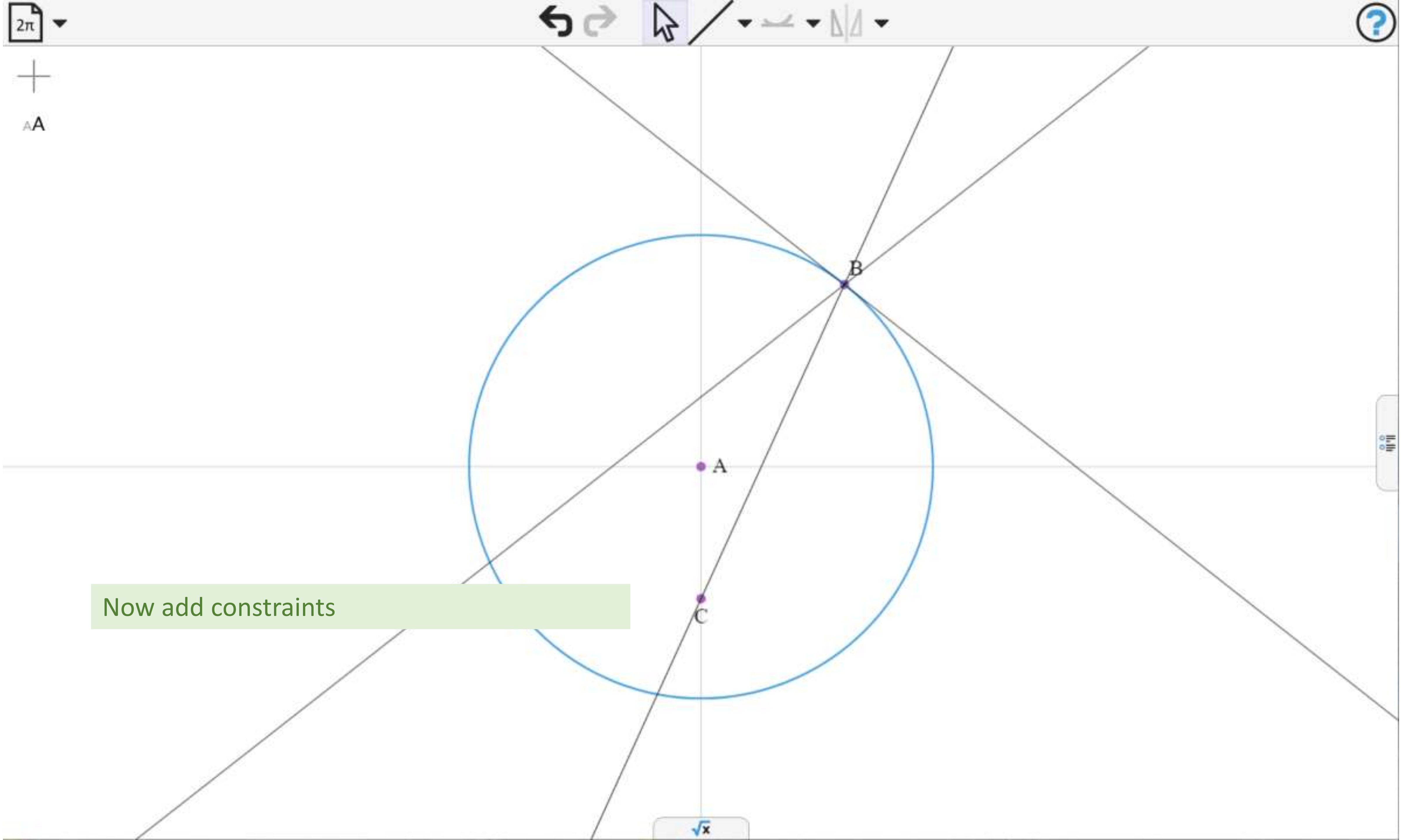
Point C on the y-axis, and a line joining C to B.



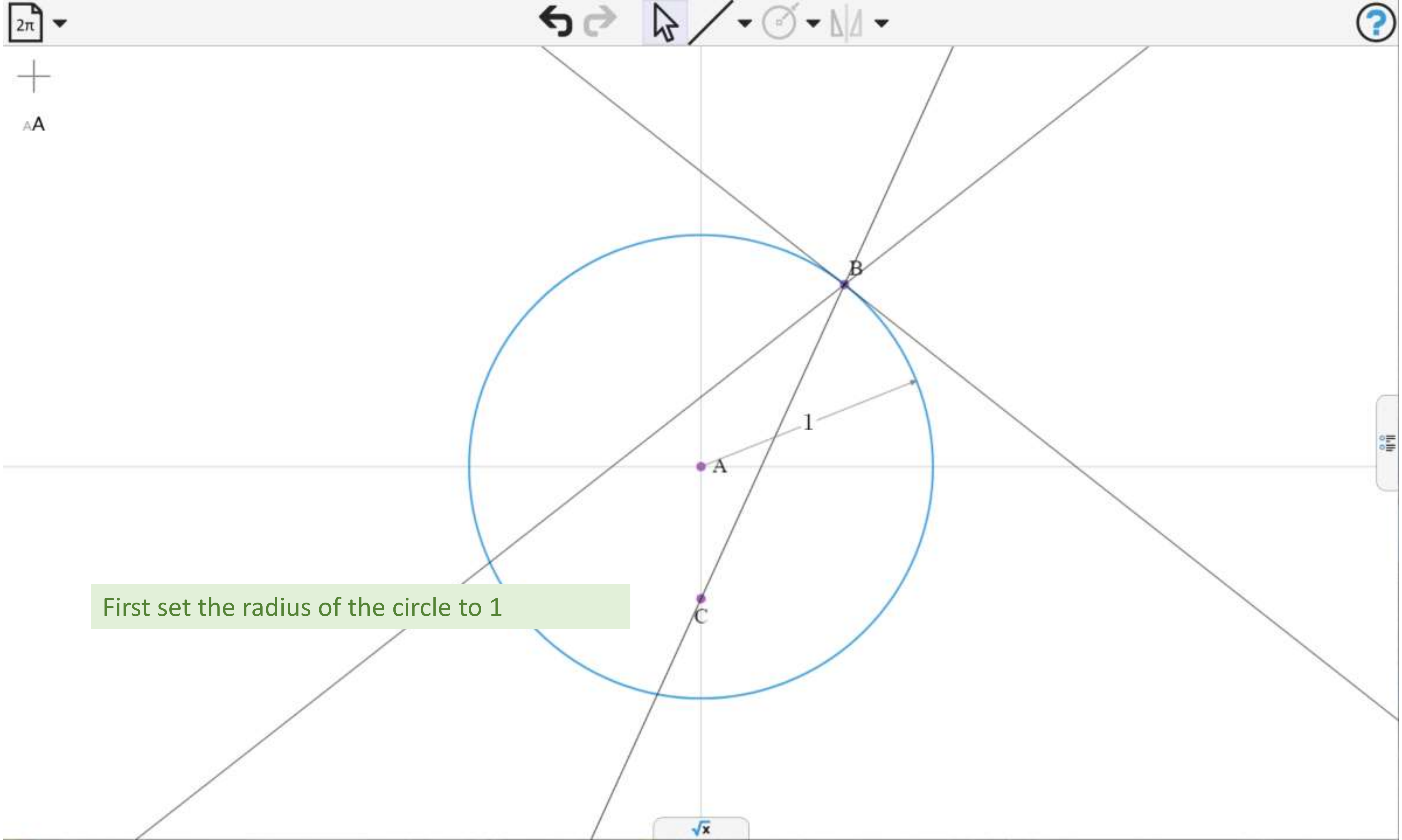
Select B and the circle and use the tangent tool.



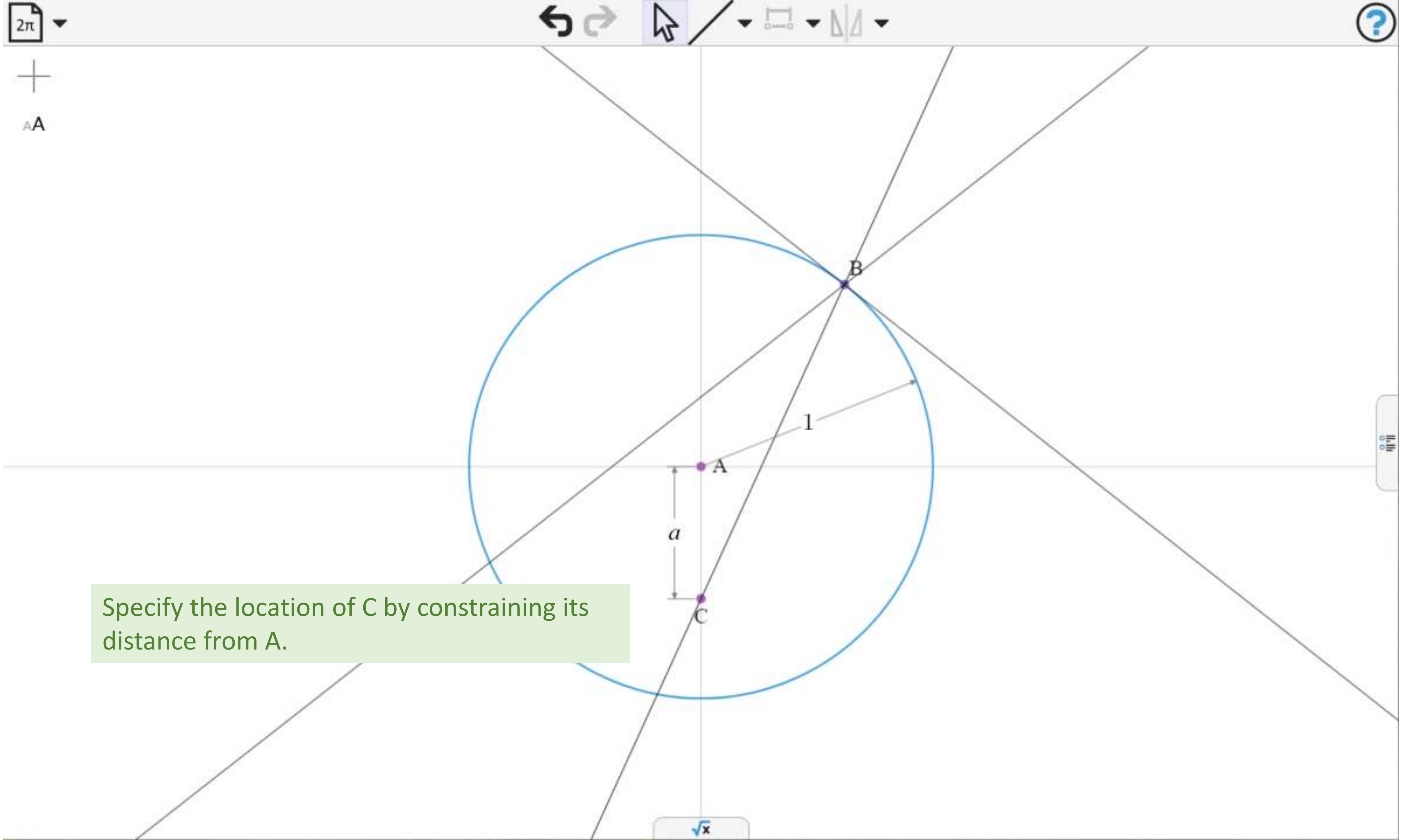
Now select the line BC and choose the reflection tool, then select the tangent as the axis of the reflection.



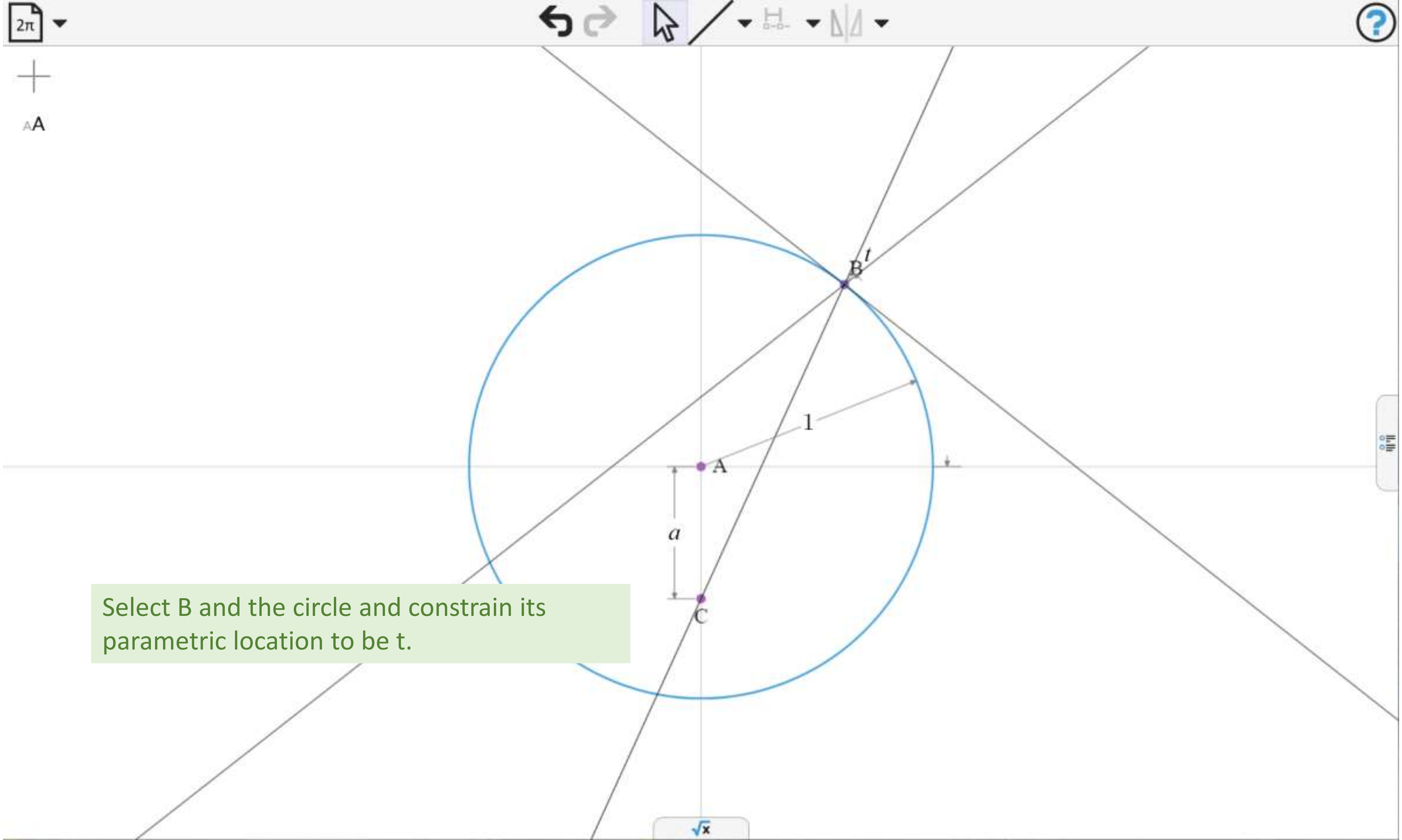
Now add constraints



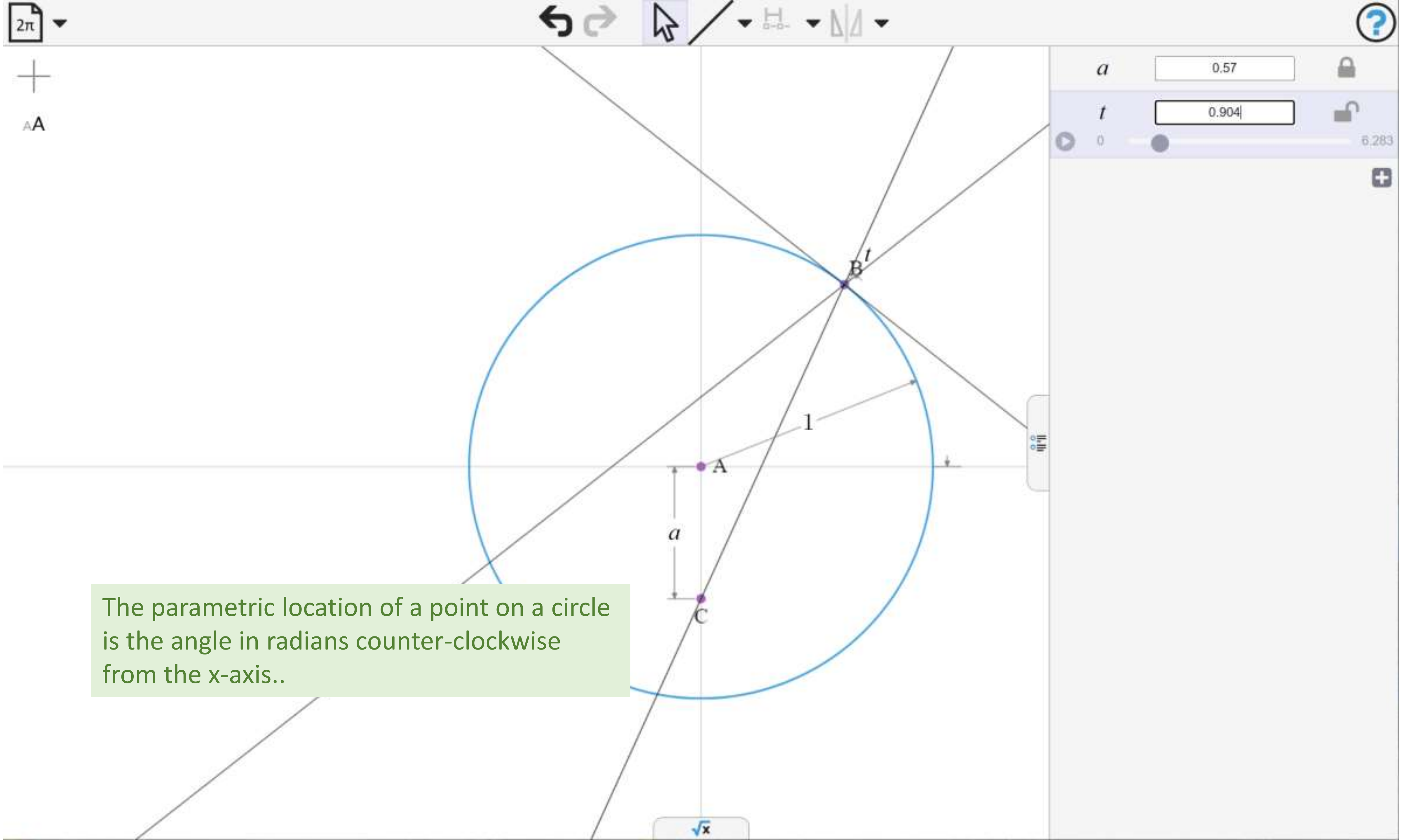
First set the radius of the circle to 1



Specify the location of C by constraining its distance from A.

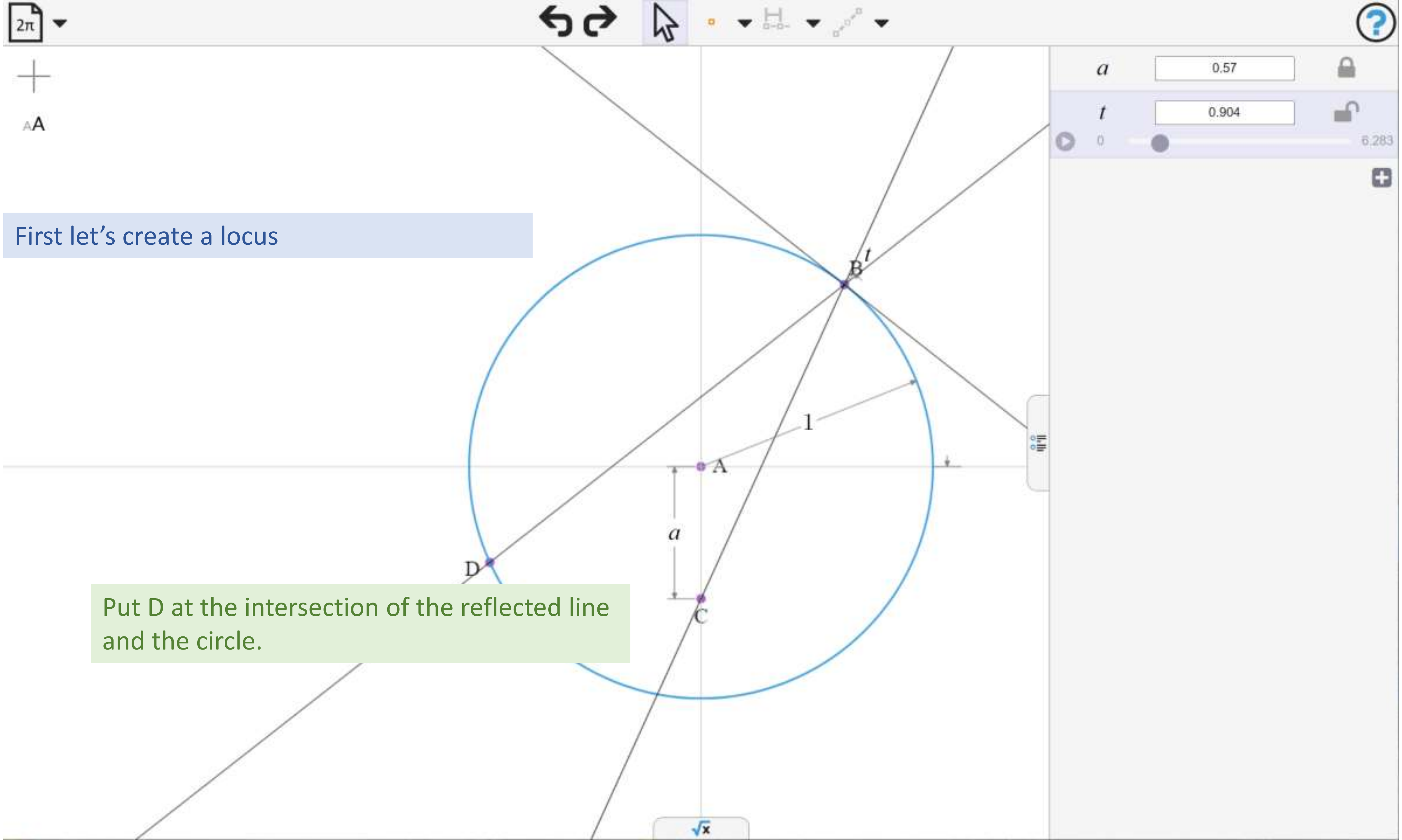


Select B and the circle and constrain its parametric location to be t.

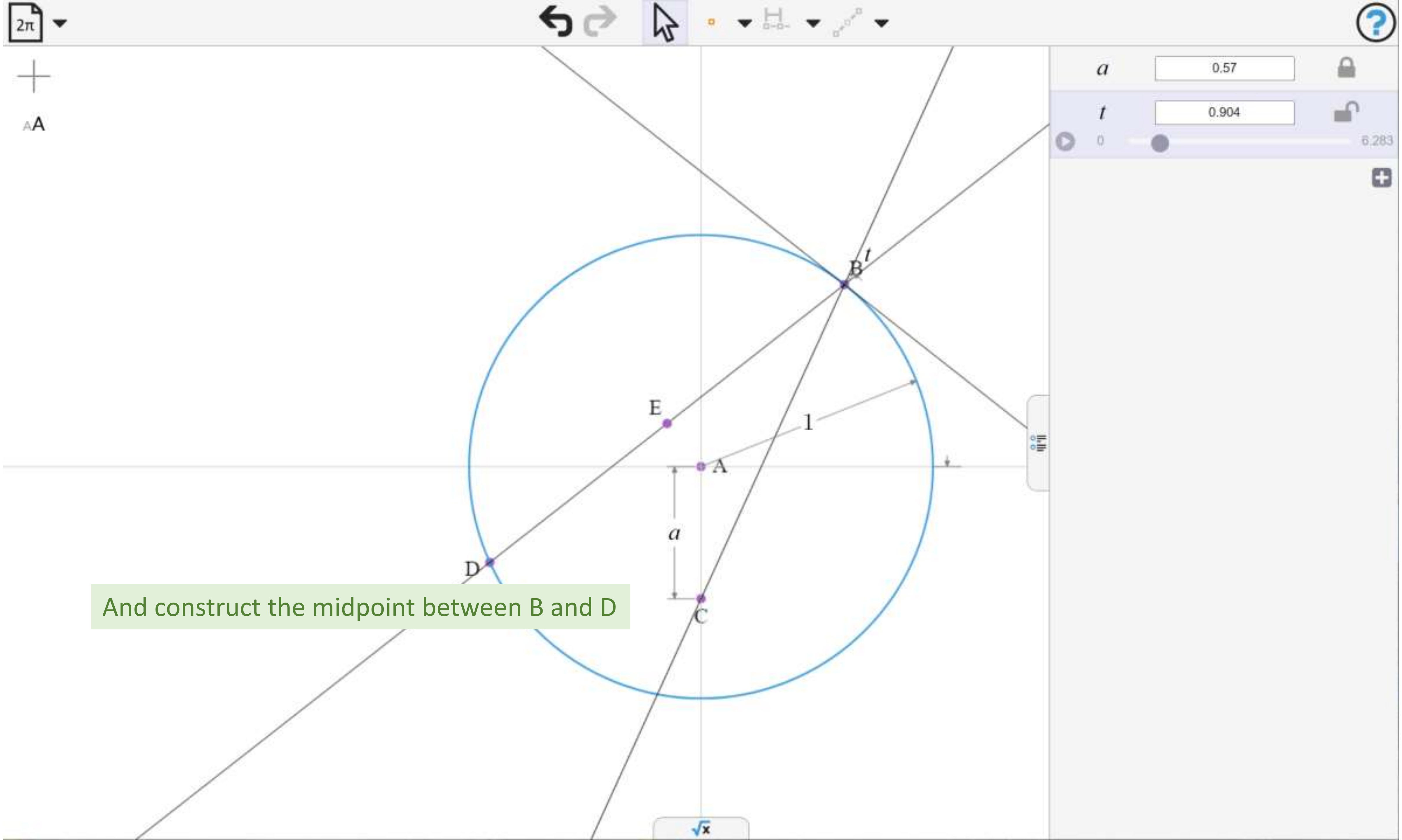


The parametric location of a point on a circle is the angle in radians counter-clockwise from the x-axis..

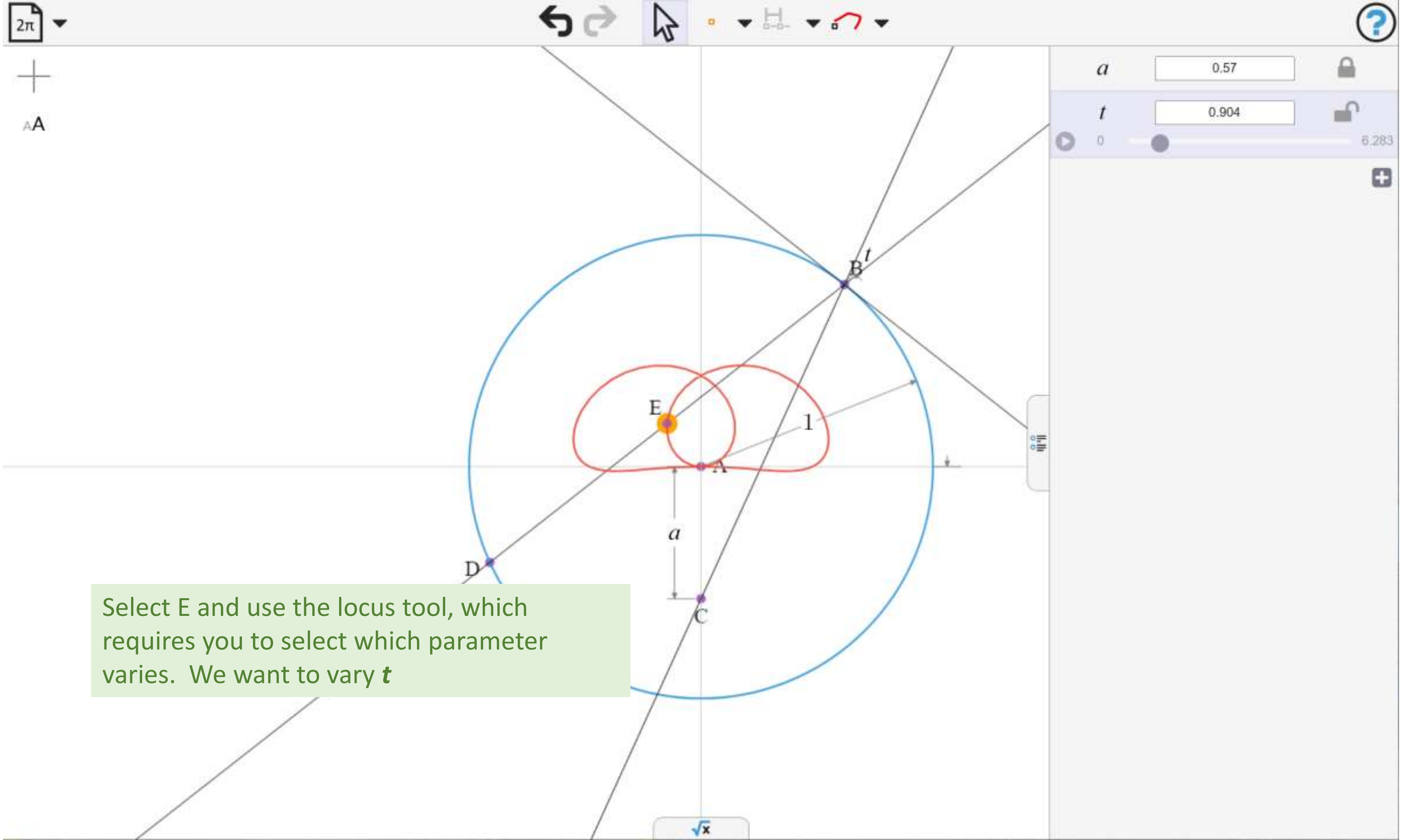
First let’s create a locus



Put D at the intersection of the reflected line and the circle.

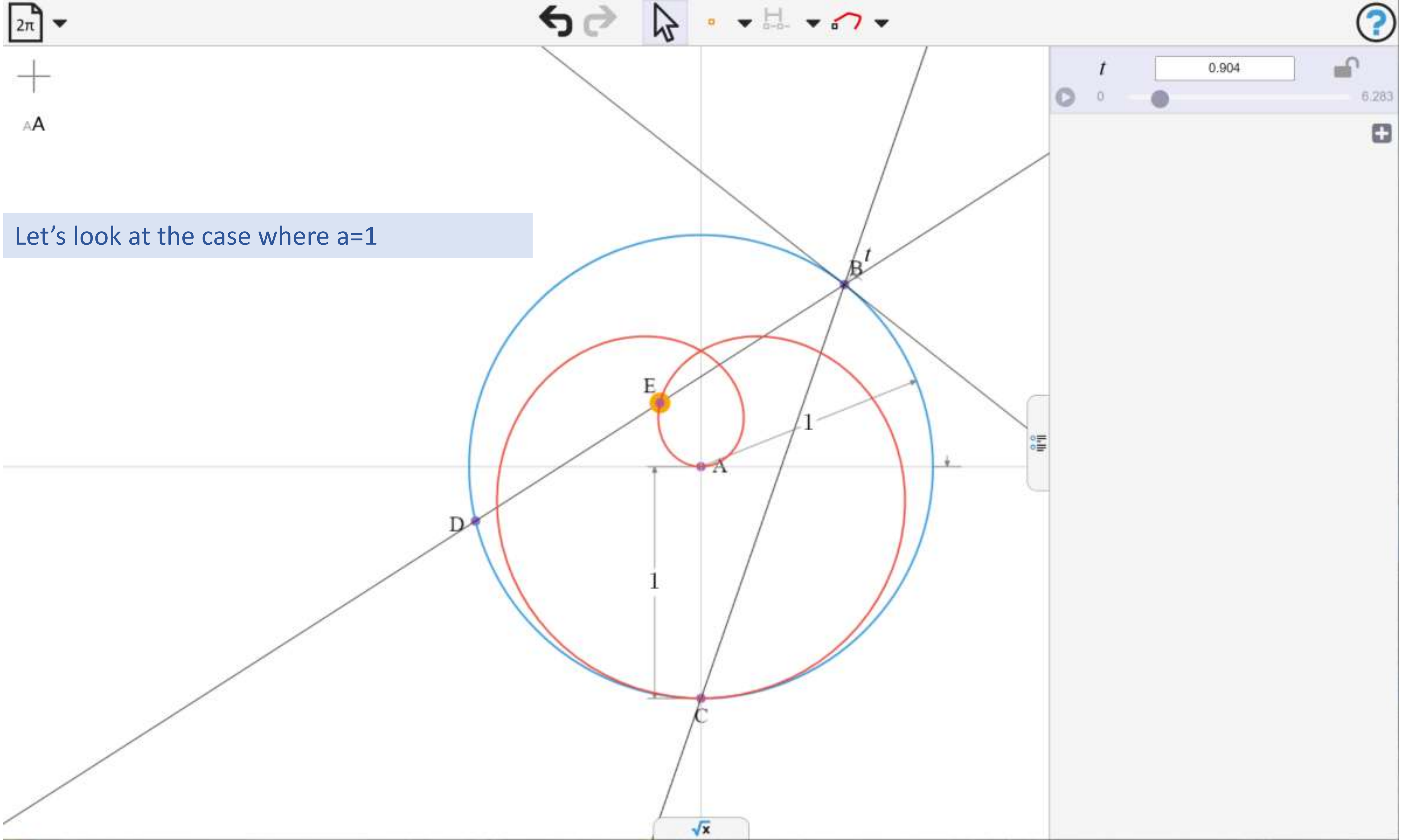


And construct the midpoint between B and D



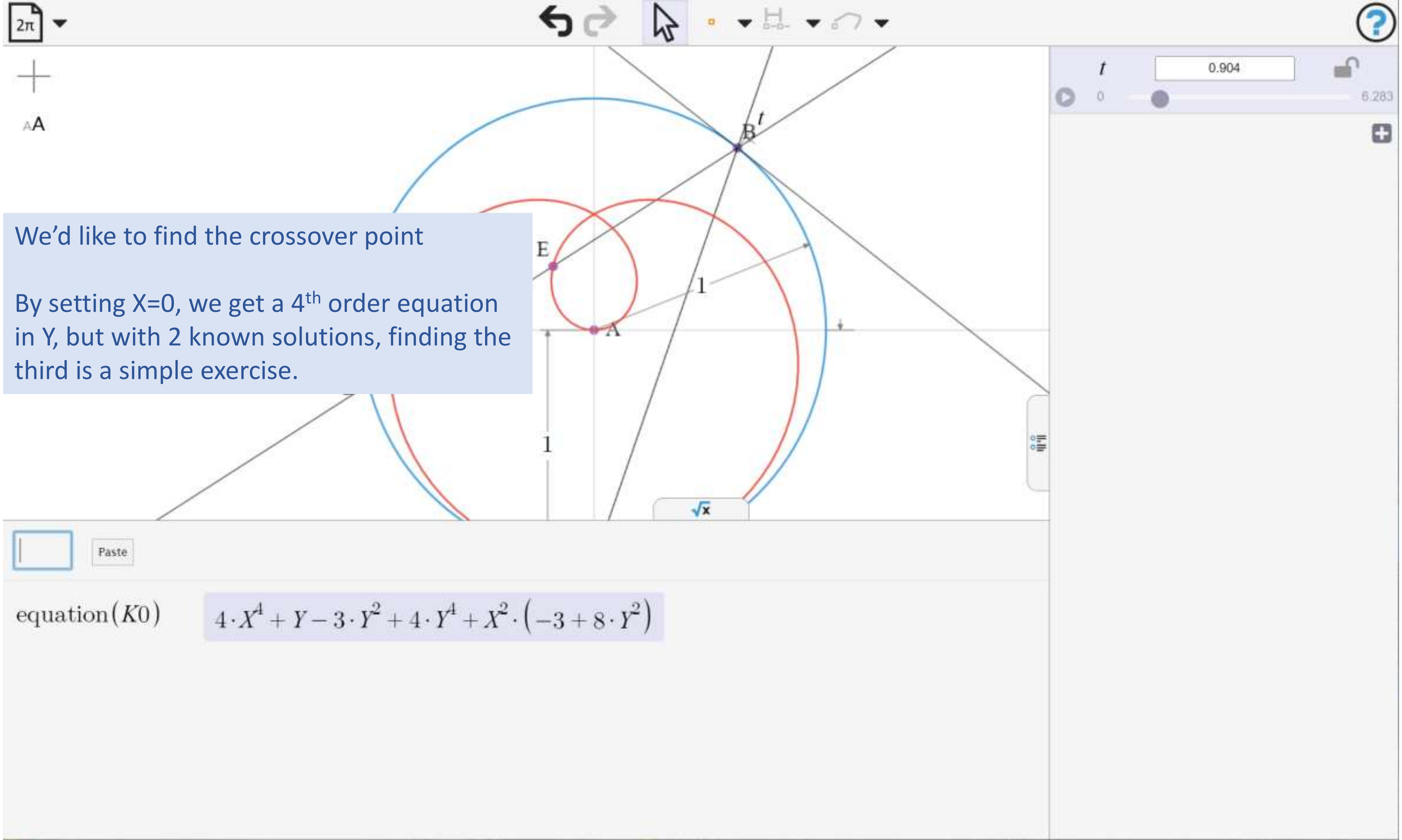
Select E and use the locus tool, which requires you to select which parameter varies. We want to vary ***t***

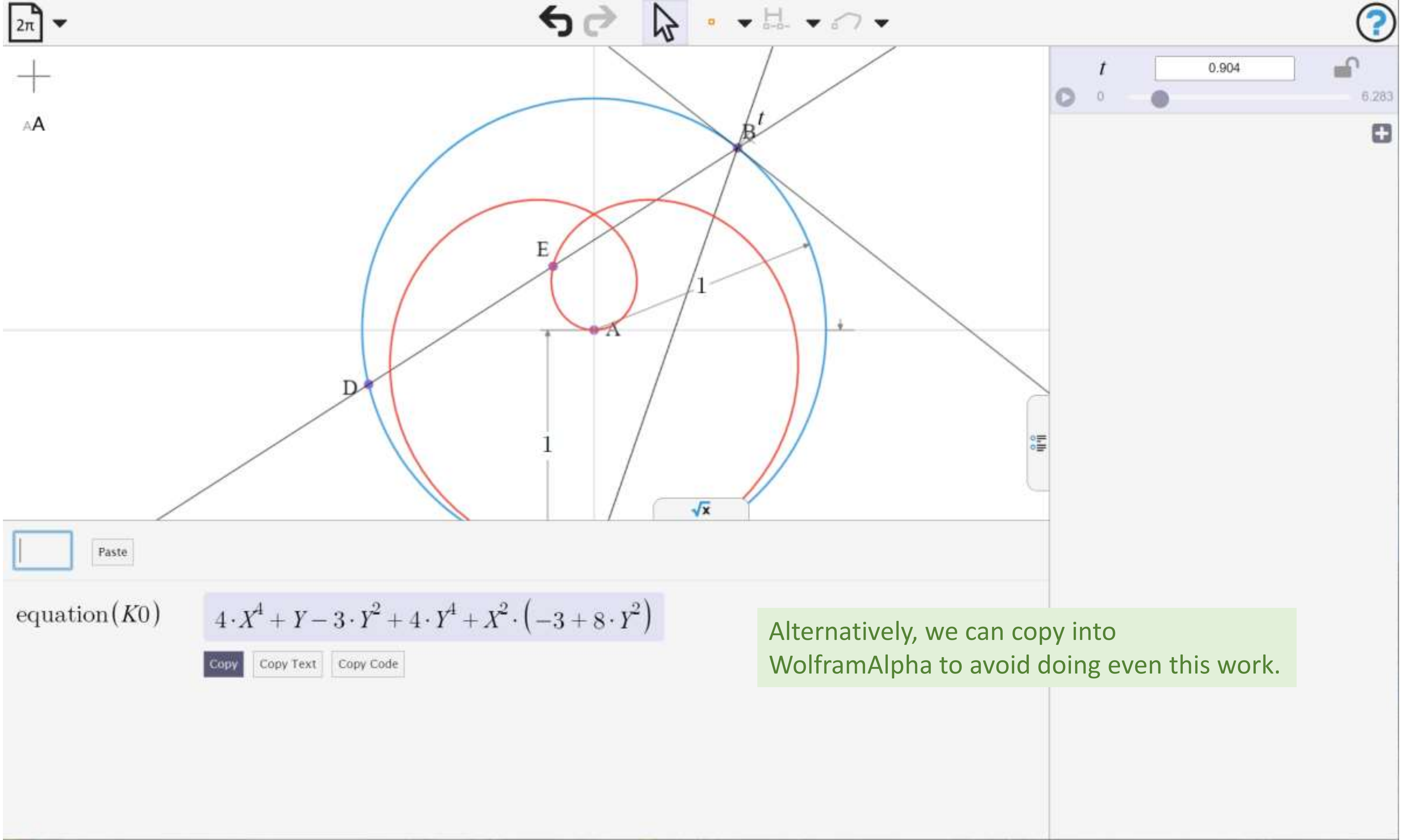
Let’s look at the case where a=1



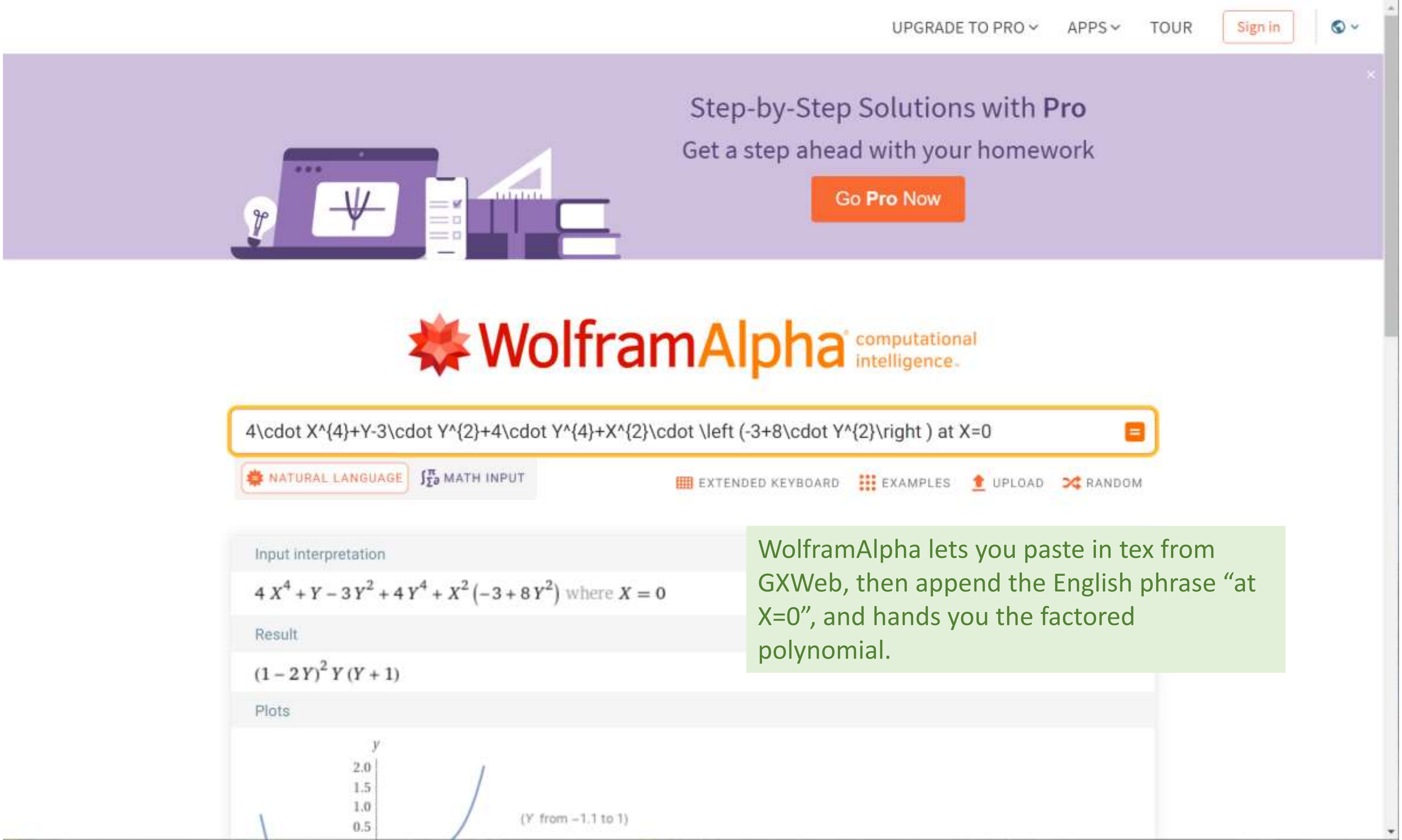
We'd like to find the crossover point

By setting X=0, we get a 4th order equation in Y, but with 2 known solutions, finding the third is a simple exercise.

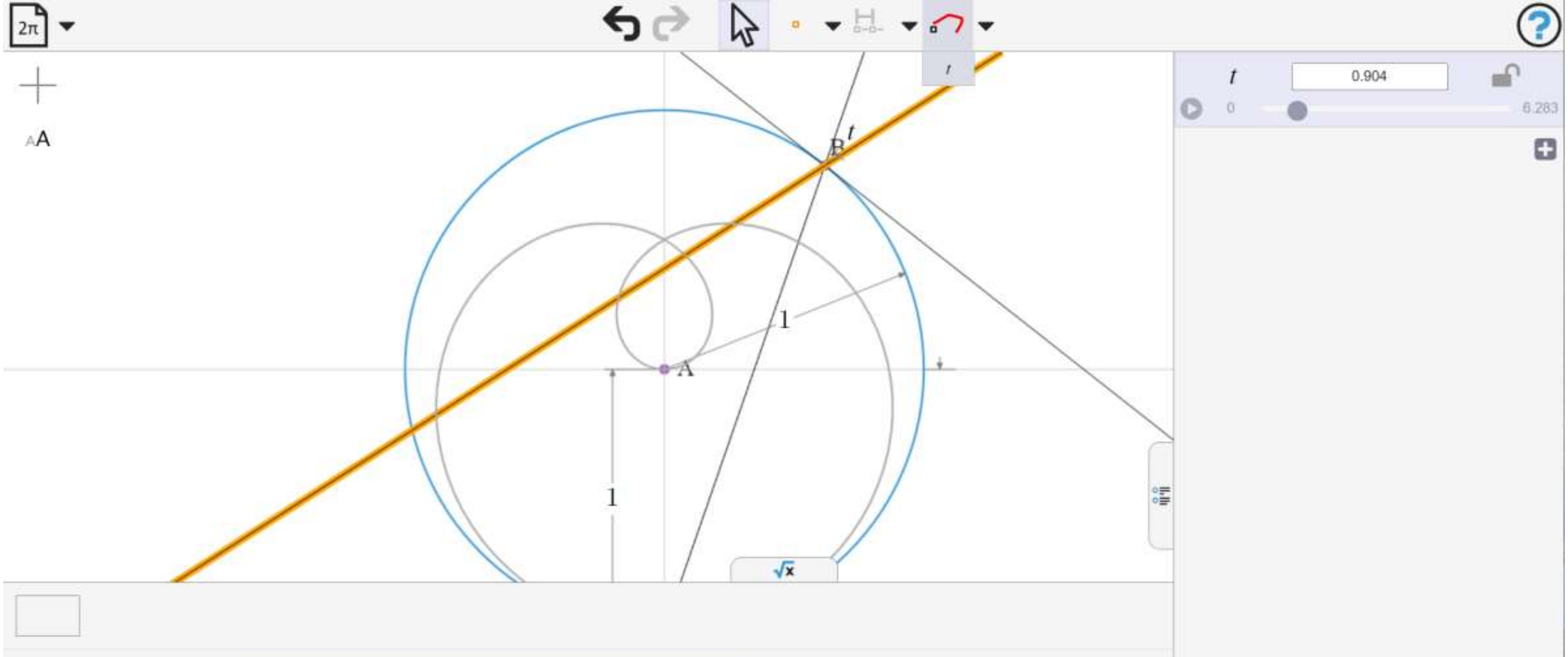




Alternatively, we can copy into WolframAlpha to avoid doing even this work.

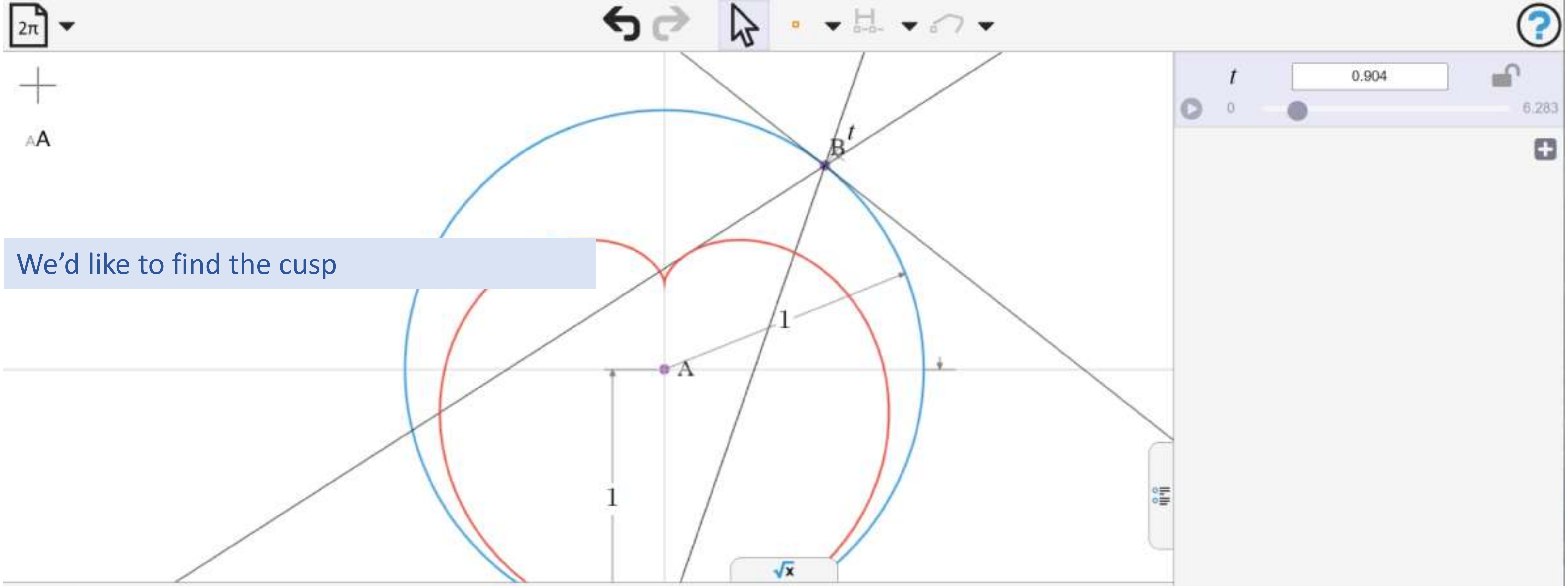


WolframAlpha lets you paste in tex from GXWeb, then append the English phrase “at X=0”, and hands you the factored polynomial.



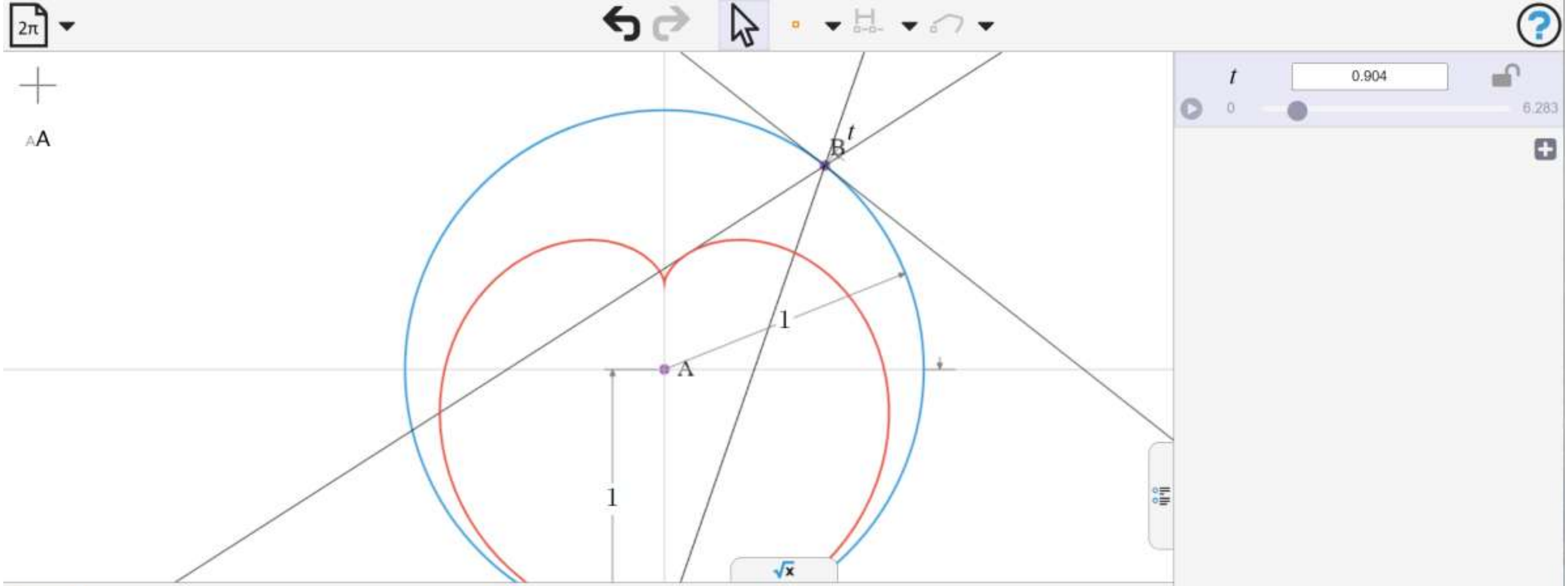
$$4 \cdot X^4 + Y - 3 \cdot Y^2 + 4 \cdot Y^4 + X^2 \cdot \left(-3 + 8 \cdot Y^2\right)$$

For an envelope, we use the locus tool with a line selected.



We'd like to find the cusp

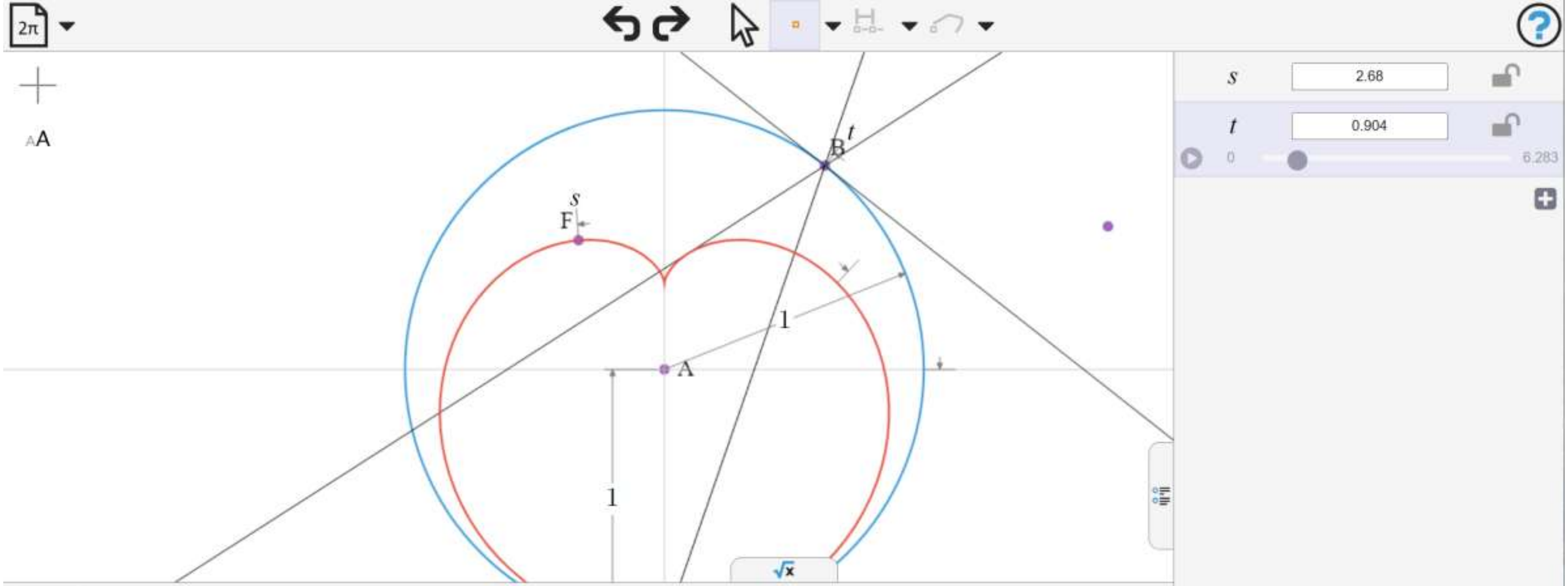
equation$(K0)$ $4 \cdot X^4 + Y - 3 \cdot Y^2 + 4 \cdot Y^4 + X^2 \cdot (-3 + 8 \cdot Y^2)$



We could proceed as before using the implicit equation, but here is a different way.

equation(K1) $-1-18\cdot X^2+27\cdot X^4+8\cdot Y-18\cdot Y^2+54\cdot X^2\cdot Y^2+27\cdot Y^4$

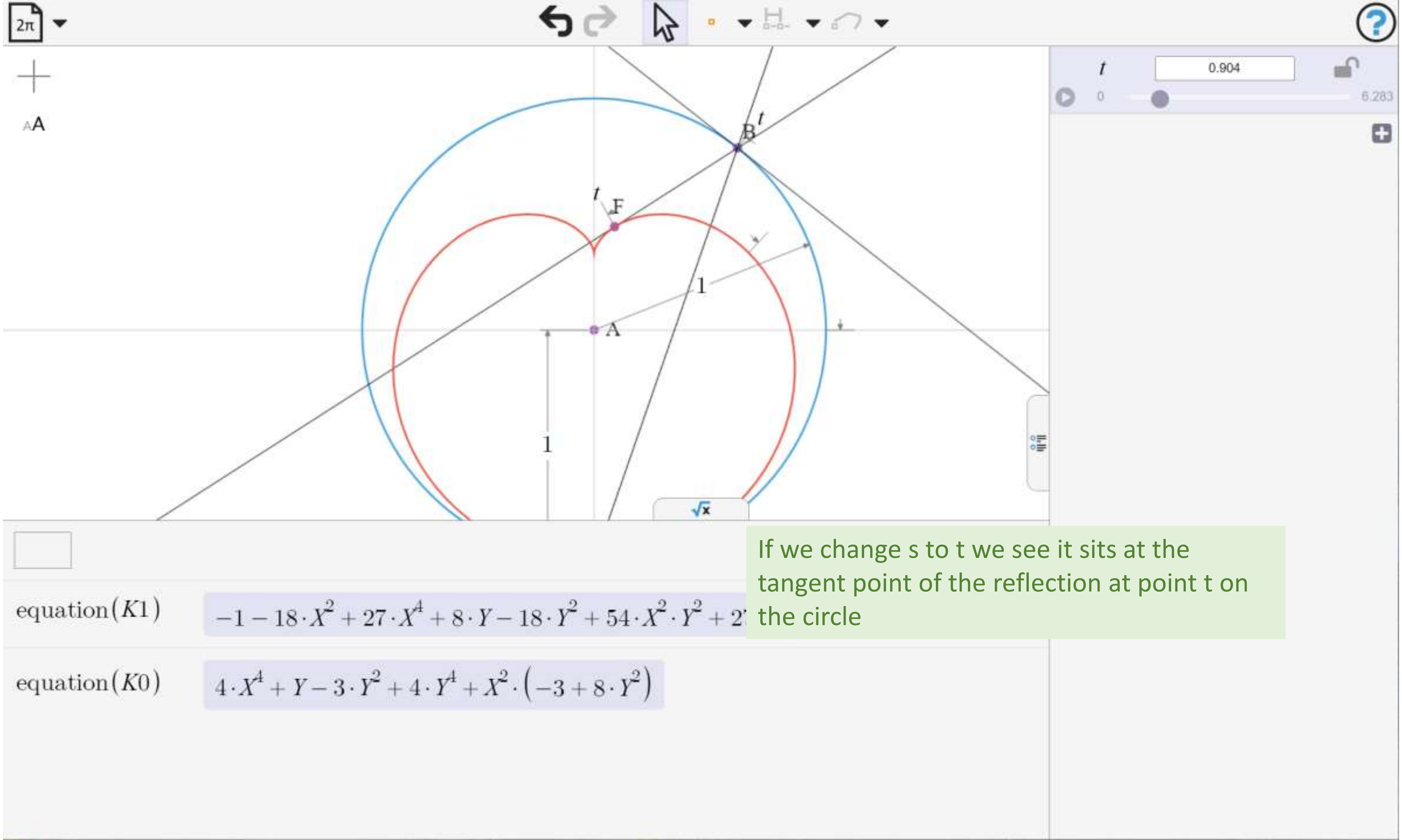
equation(K0) $4\cdot X^4+Y-3\cdot Y^2+4\cdot Y^4+X^2\cdot\left(-3+8\cdot Y^2\right)$



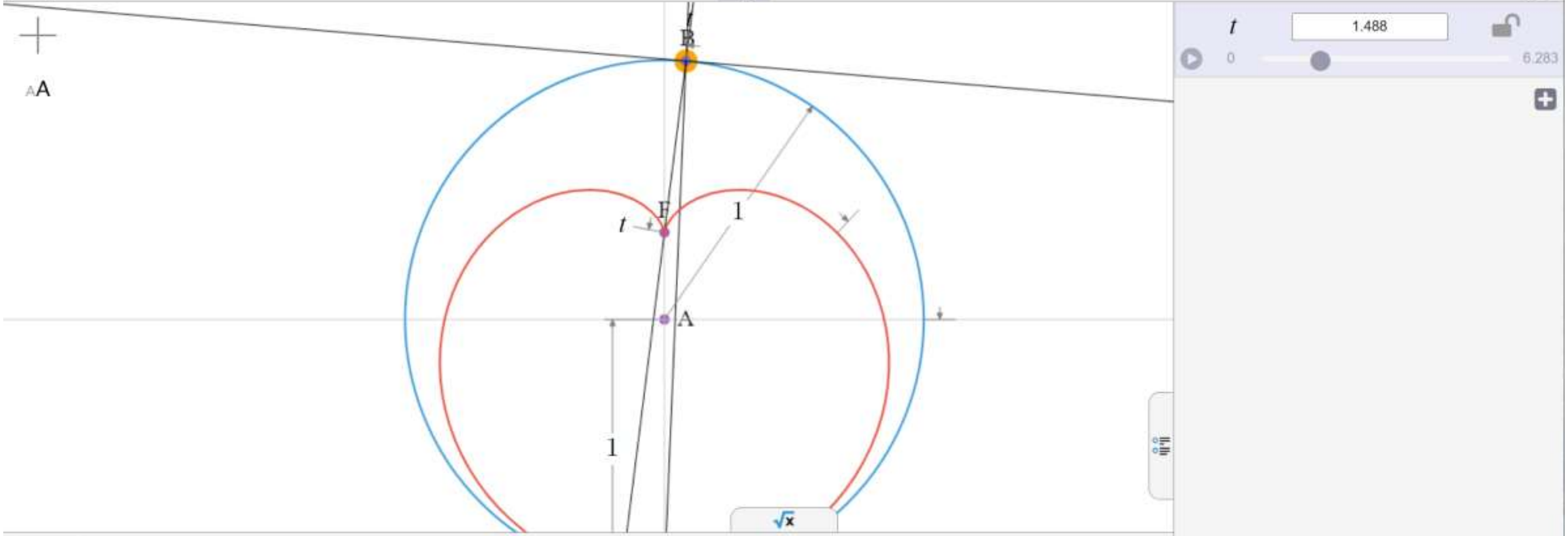
First we put point F on the curve, GXWeb adds a parametric location s.

equation(K1) $-1-18\cdot X^2+27\cdot X^4+8\cdot Y-18\cdot Y^2+54\cdot X^2\cdot Y^2+27\cdot Y^4$

equation(K0) $4\cdot X^4+Y-3\cdot Y^2+4\cdot Y^4+X^2\cdot\left(-3+8\cdot Y^2\right)$



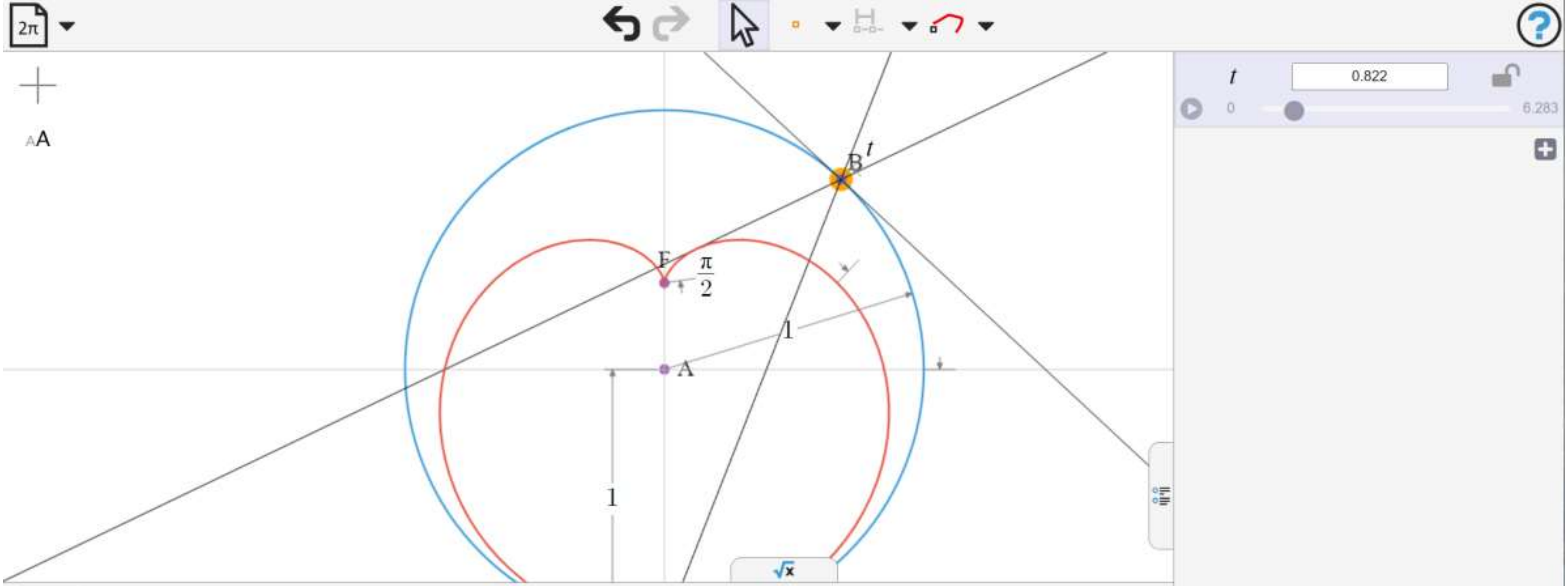
If we change s to t we see it sits at the tangent point of the reflection at point t on the circle



Clearly the cusp corresponds to t = π/2

equation(K1) $-1-18\cdot X^2+27\cdot X^4+8\cdot Y-18\cdot Y^2+54\cdot X^2\cdot Y^2+27\cdot Y^4$

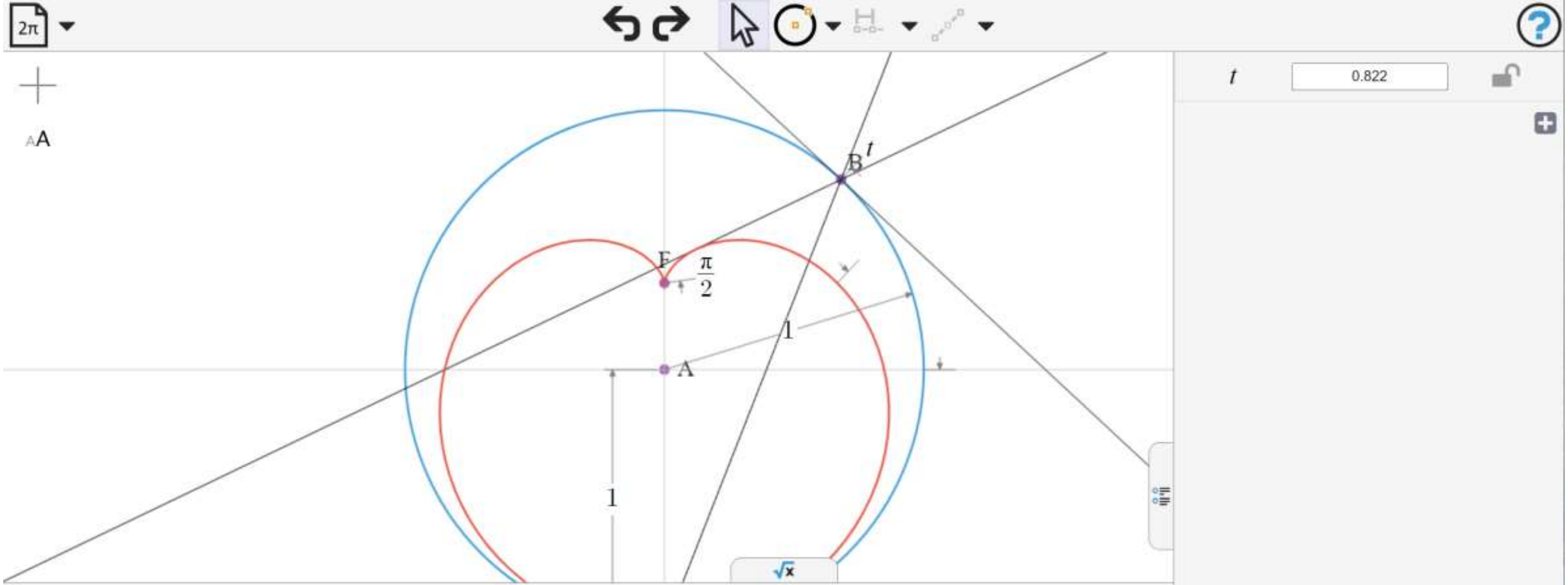
equation(K0) $4\cdot X^4+Y-3\cdot Y^2+4\cdot Y^4+X^2\cdot\left(-3+8\cdot Y^2\right)$



Set the parametric location of F to π/2
And it sits on the cusp

equation$(K1)$ $-1 - 18 \cdot X^2 + 27 \cdot X^4 + 8 \cdot Y - 18 \cdot Y^2 + 54 \cdot X^2 \cdot Y^2 + 27 \cdot Y^4$

equation$(K0)$ $4 \cdot X^4 + Y - 3 \cdot Y^2 + 4 \cdot Y^4 + X^2 \cdot \left(-3 + 8 \cdot Y^2\right)$

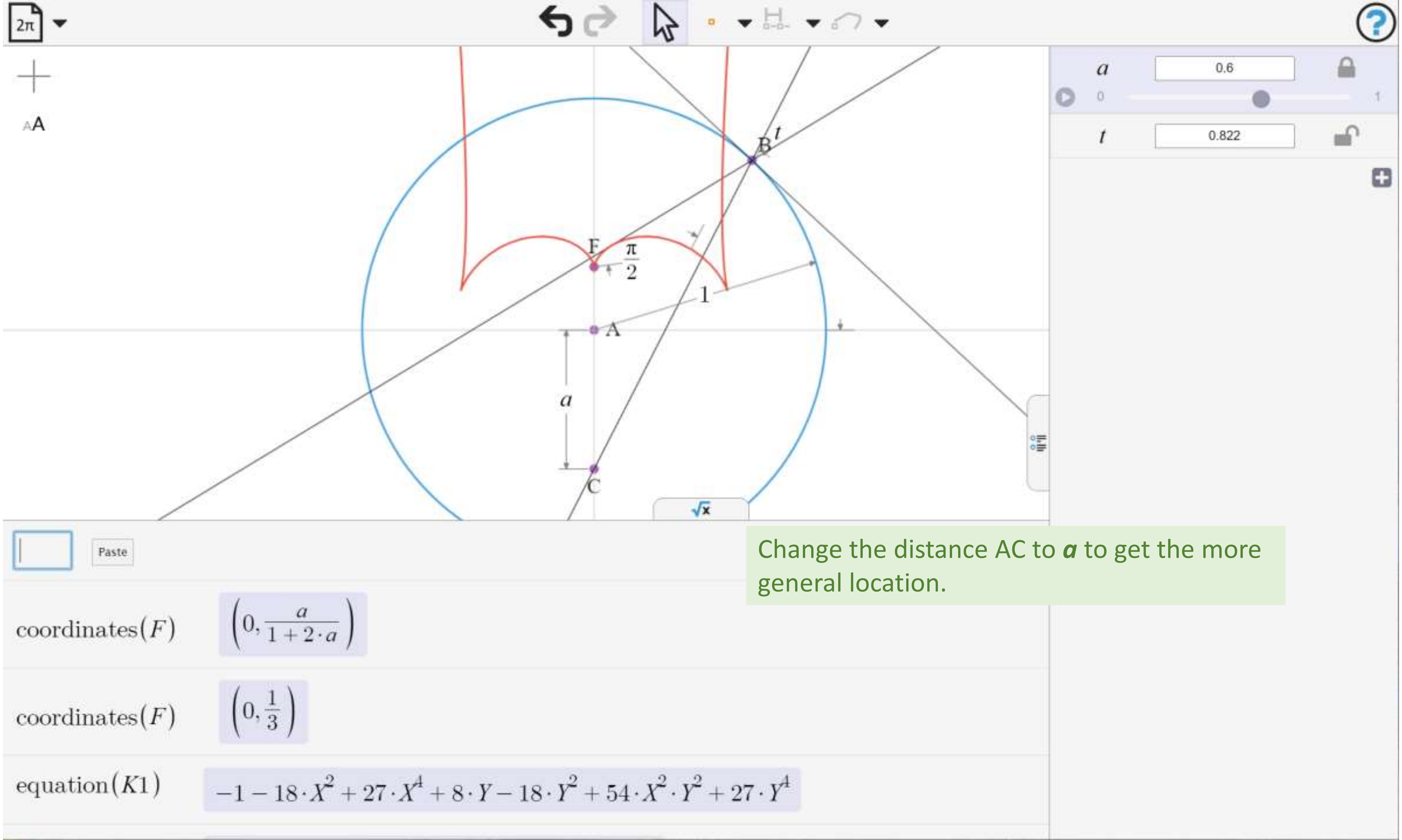


Asking for its coordinates gives the location of the cusp

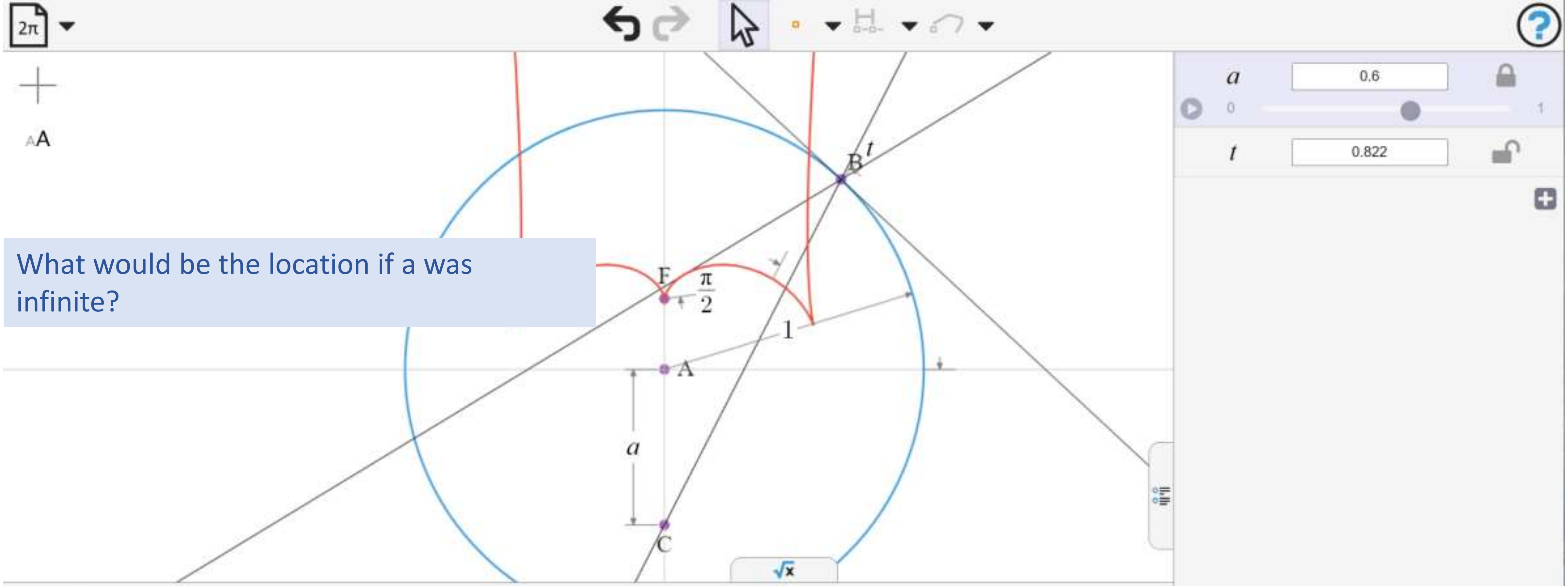
coordinates(F) $\left(0, \frac{1}{3}\right)$

equation$(K1)$ $-1 - 18 \cdot X^2 + 27 \cdot X^4 + 8 \cdot Y - 18 \cdot Y^2 + 54 \cdot X^2 \cdot Y^2 + 27 \cdot Y^4$

equation$(K0)$ $4 \cdot X^4 + Y - 3 \cdot Y^2 + 4 \cdot Y^4 + X^2 \cdot \left(-3 + 8 \cdot Y^2\right)$



Change the distance AC to ***a*** to get the more general location.



What would be the location if a was infinite?

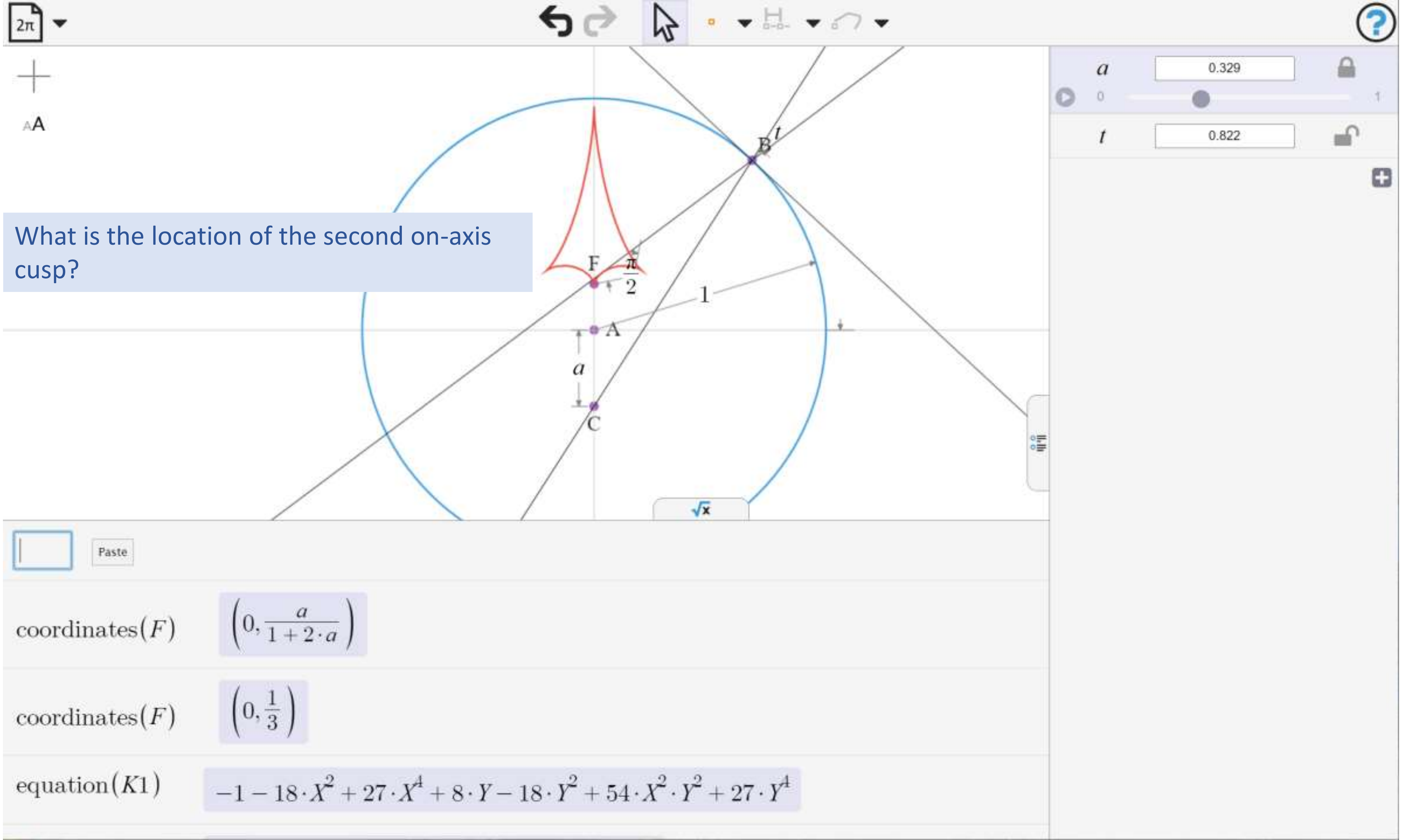
Change the distance AC to ***a*** to get the more general location.

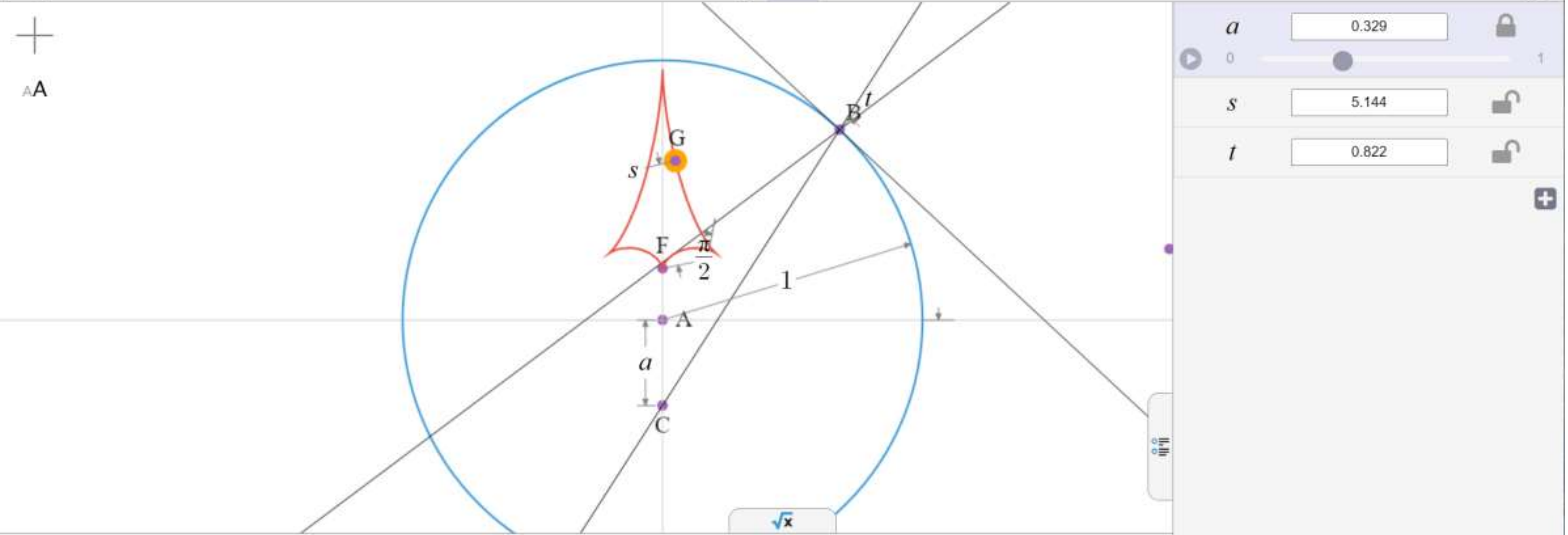
coordinates(F) $\left(0, \frac{a}{1+2\cdot a}\right)$

coordinates(F) $\left(0, \frac{1}{3}\right)$

equation$(K1)$ $-1-18\cdot X^2+27\cdot X^4+8\cdot Y-18\cdot Y^2+54\cdot X^2\cdot Y^2+27\cdot Y^4$

What is the location of the second on-axis cusp?



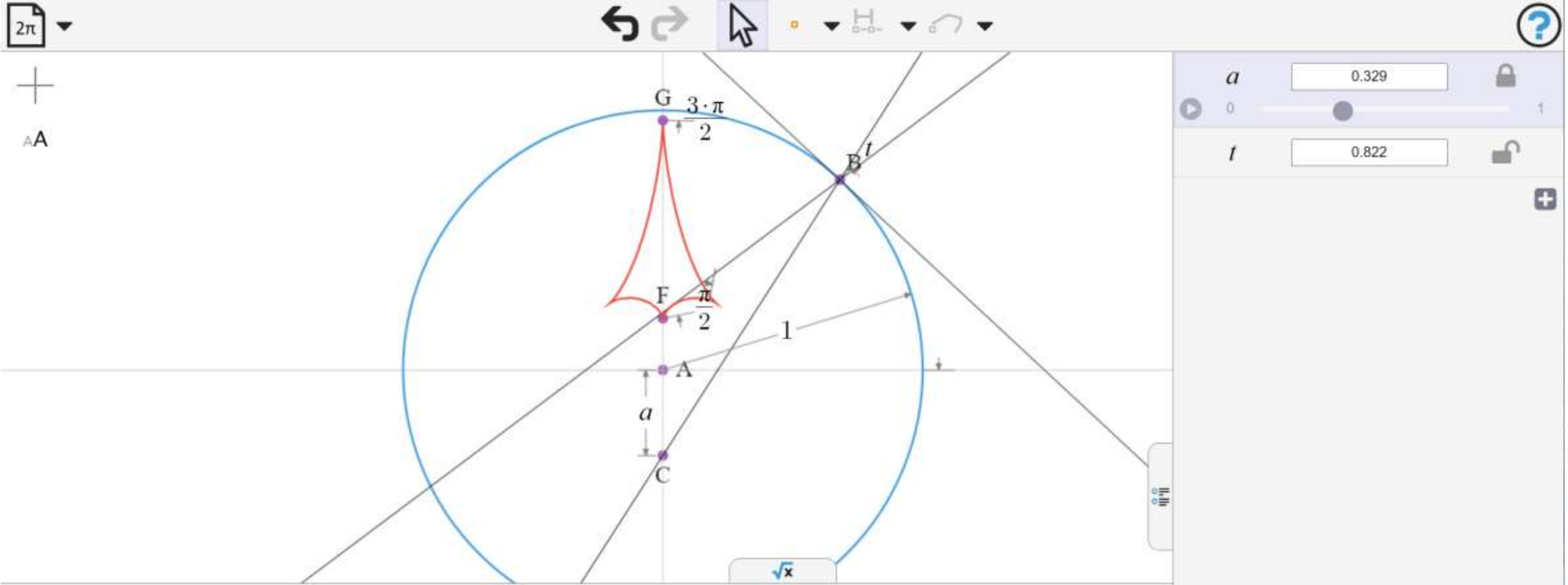


Clearly the cusp corresponds to t = 3π/2

coordinates(F) $\left(0, \frac{a}{1+2 \cdot a}\right)$

coordinates(F) $\left(0, \frac{1}{3}\right)$

equation($K1$) $-1 - 18 \cdot X^2 + 27 \cdot X^4 + 8 \cdot Y - 18 \cdot Y^2 + 54 \cdot X^2 \cdot Y^2 + 27 \cdot Y^4$

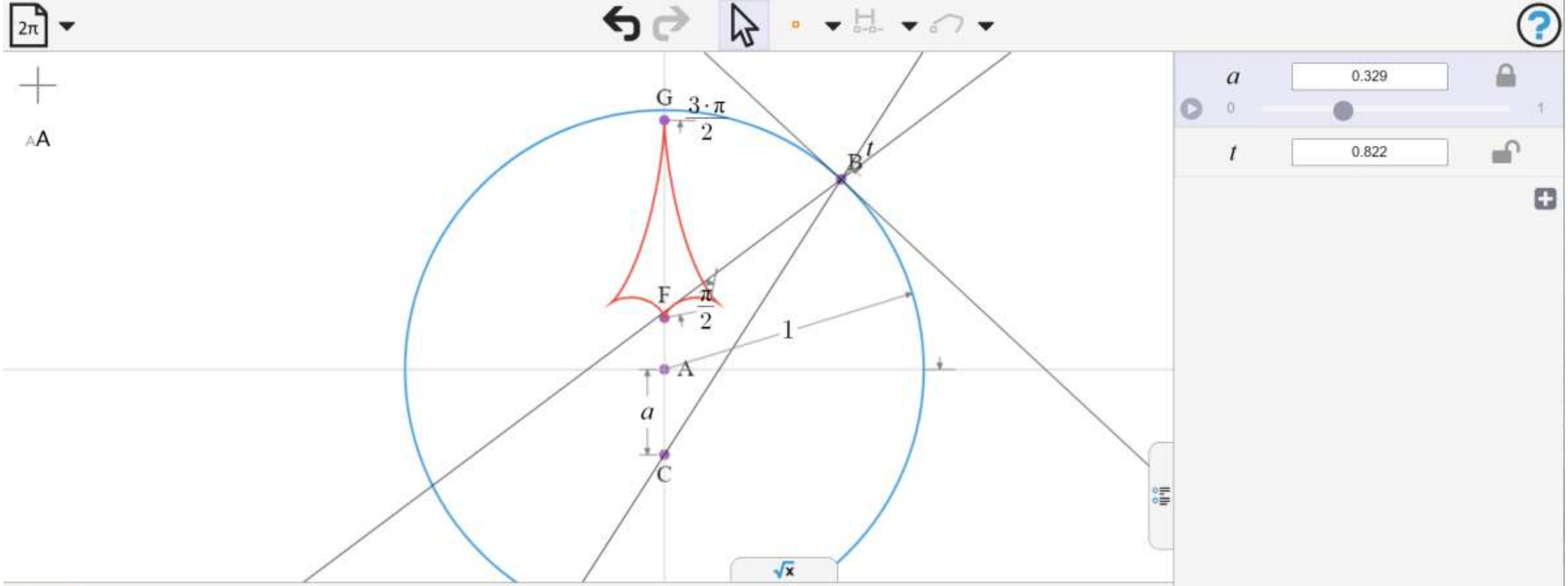


So we create a point at parametric location 3π/2

coordinates(F) $\left(0, \frac{a}{1+2\cdot a}\right)$

coordinates(F) $\left(0, \frac{1}{3}\right)$

equation$(K1)$ $-1-18\cdot X^2+27\cdot X^4+8\cdot Y-18\cdot Y^2+54\cdot X^2\cdot Y^2+27\cdot Y^4$



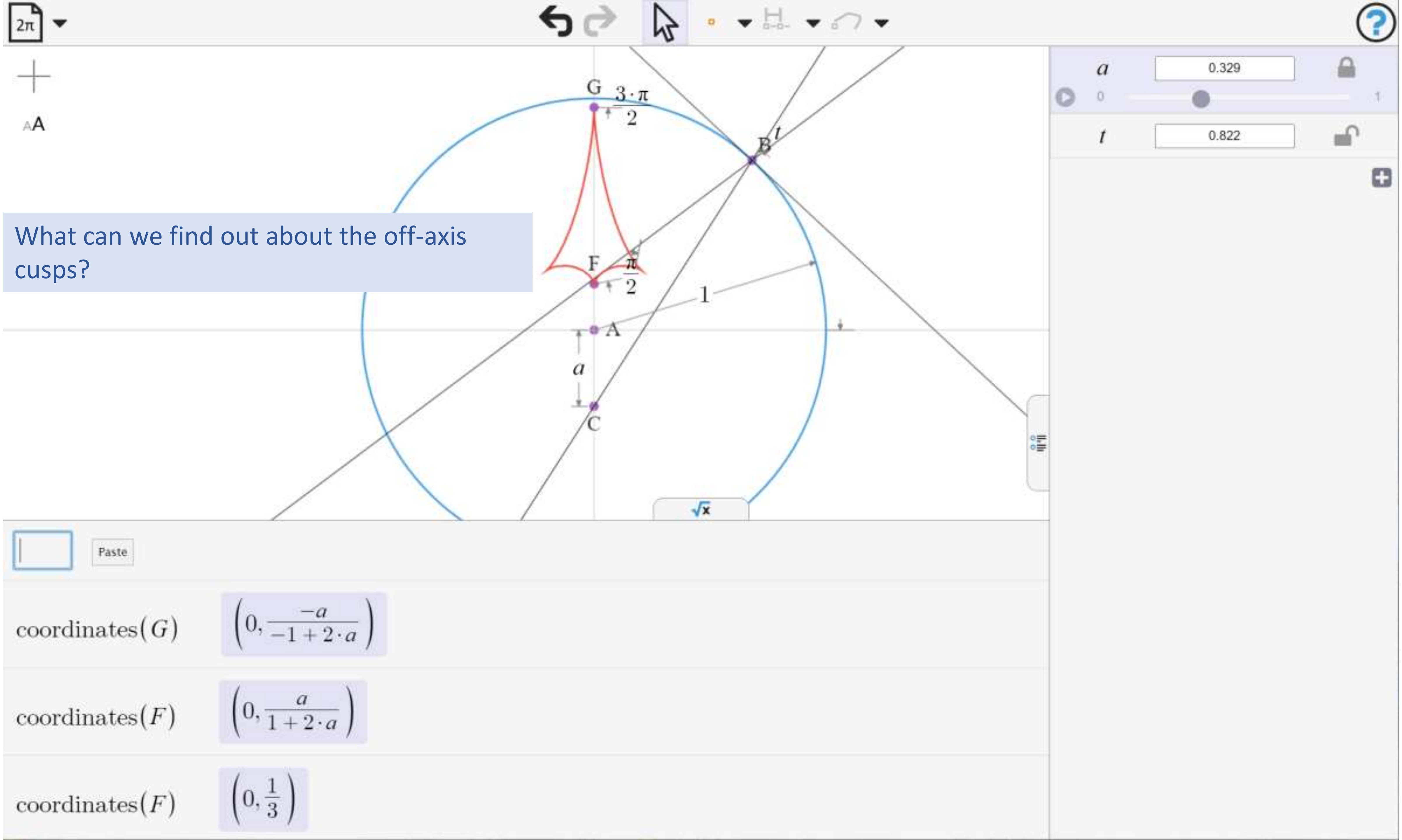
And get its coordinates

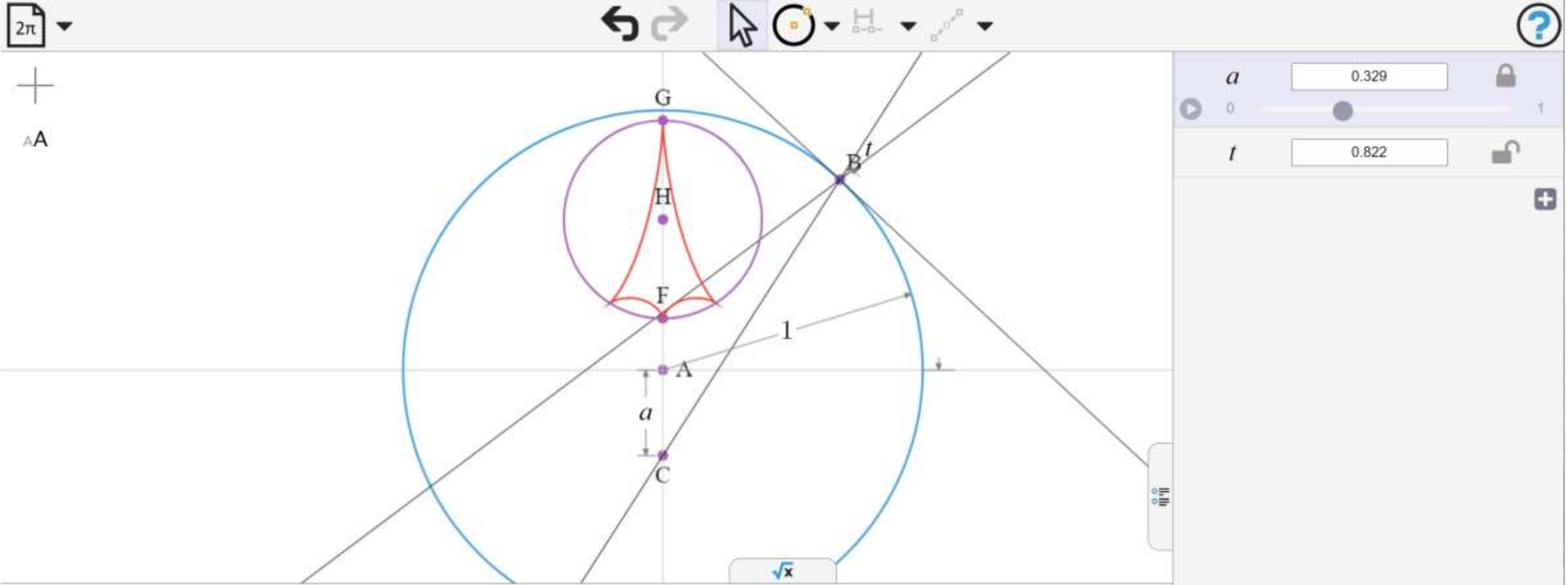
coordinates(G) $\left(0, \frac{-a}{-1+2\cdot a}\right)$

coordinates(F) $\left(0, \frac{a}{1+2\cdot a}\right)$

coordinates(F) $\left(0, \frac{1}{3}\right)$

What can we find out about the off-axis cusps?



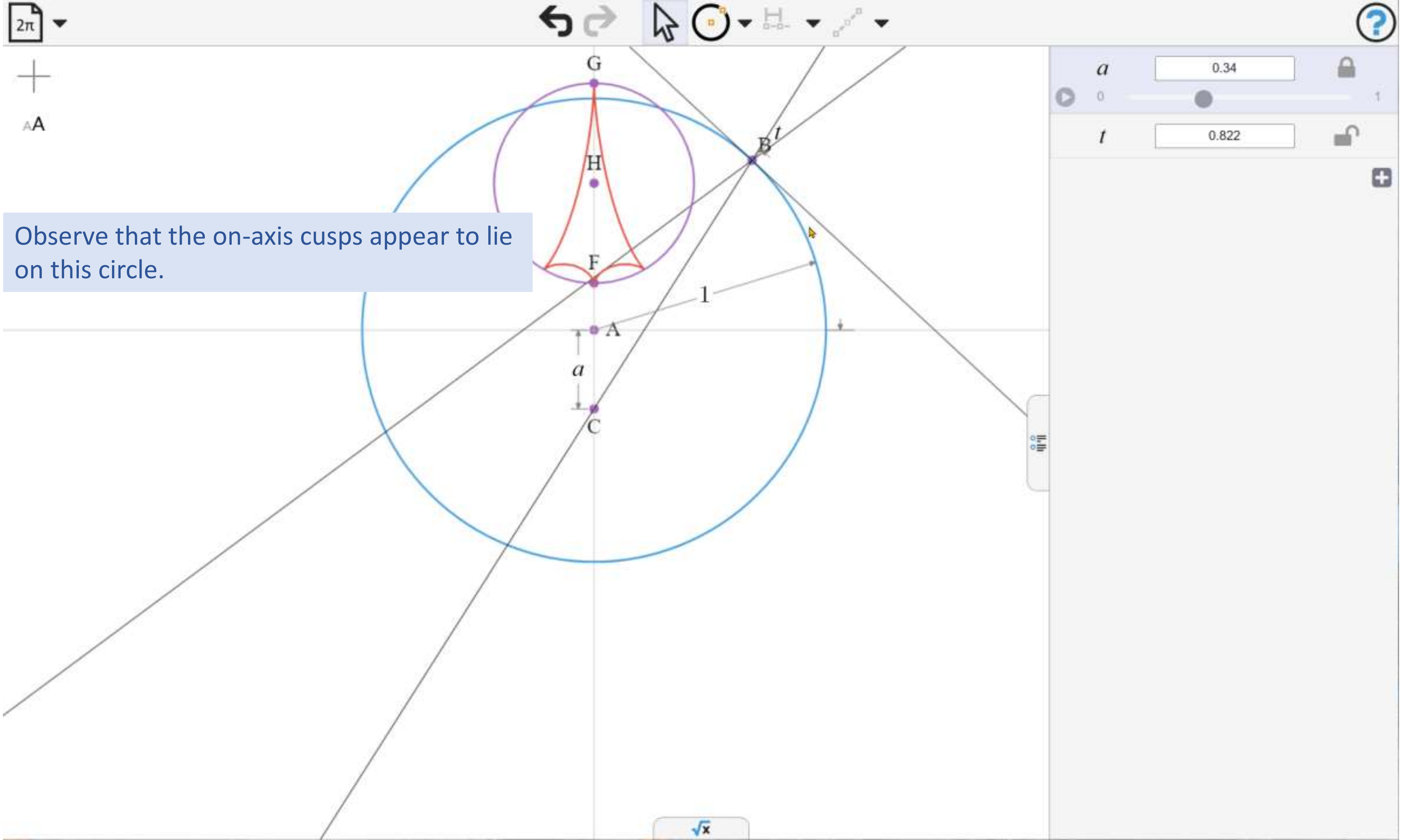


Create a circle whose diameter is the on-axis cusps.

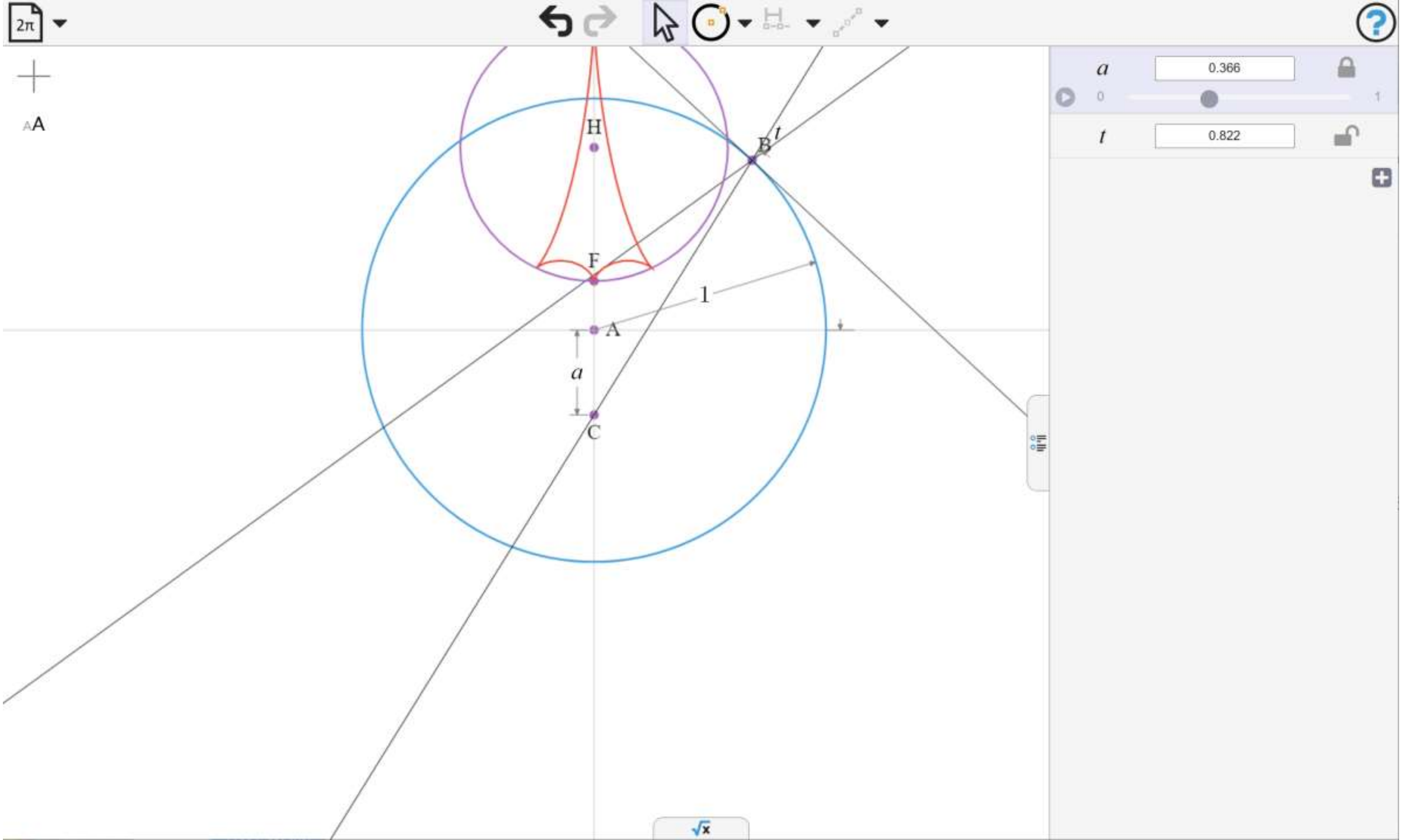
coordinates(G) $\left(0, \frac{-a}{-1+2\cdot a}\right)$

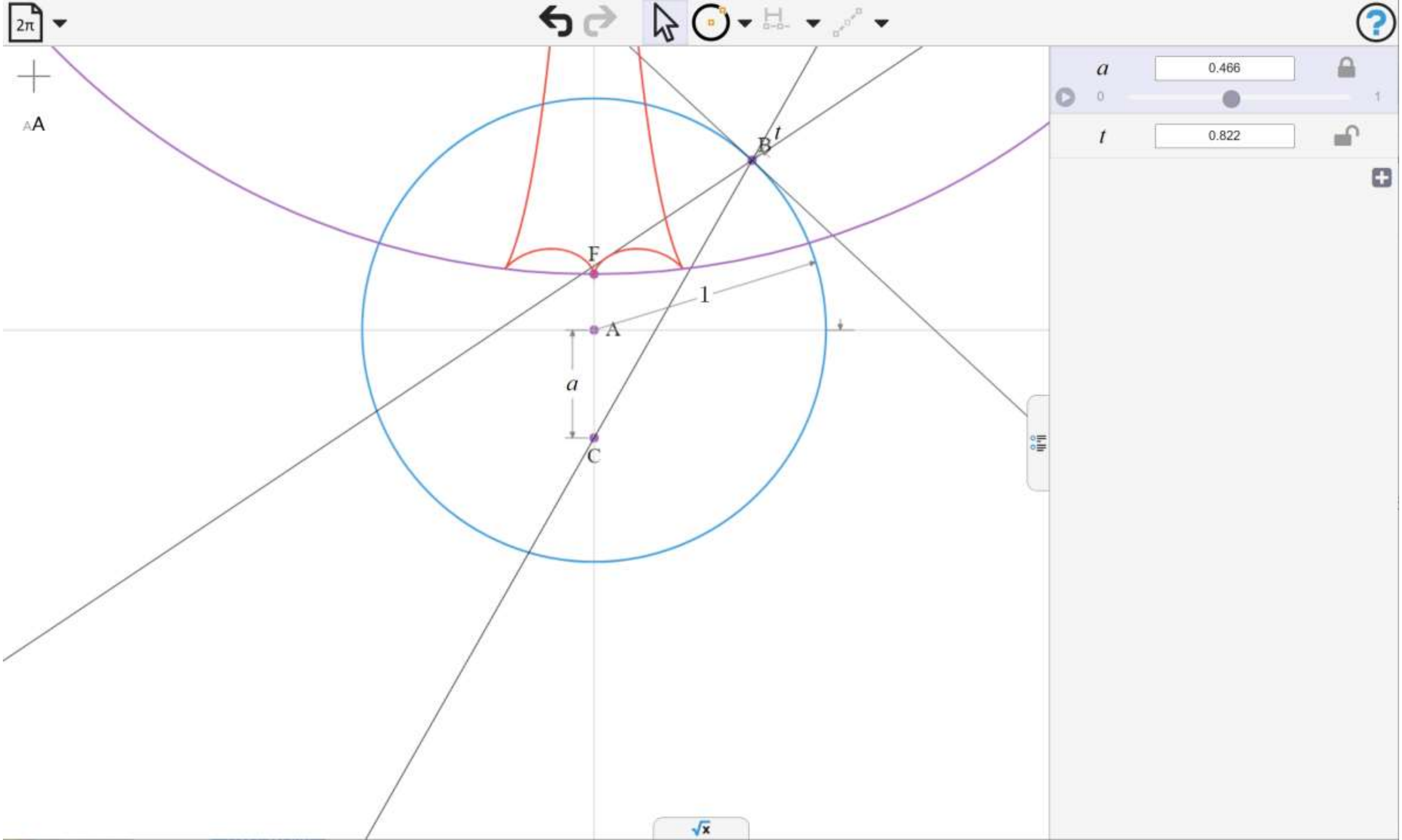
coordinates(F) $\left(0, \frac{a}{1+2\cdot a}\right)$

coordinates(F) $\left(0, \frac{1}{3}\right)$

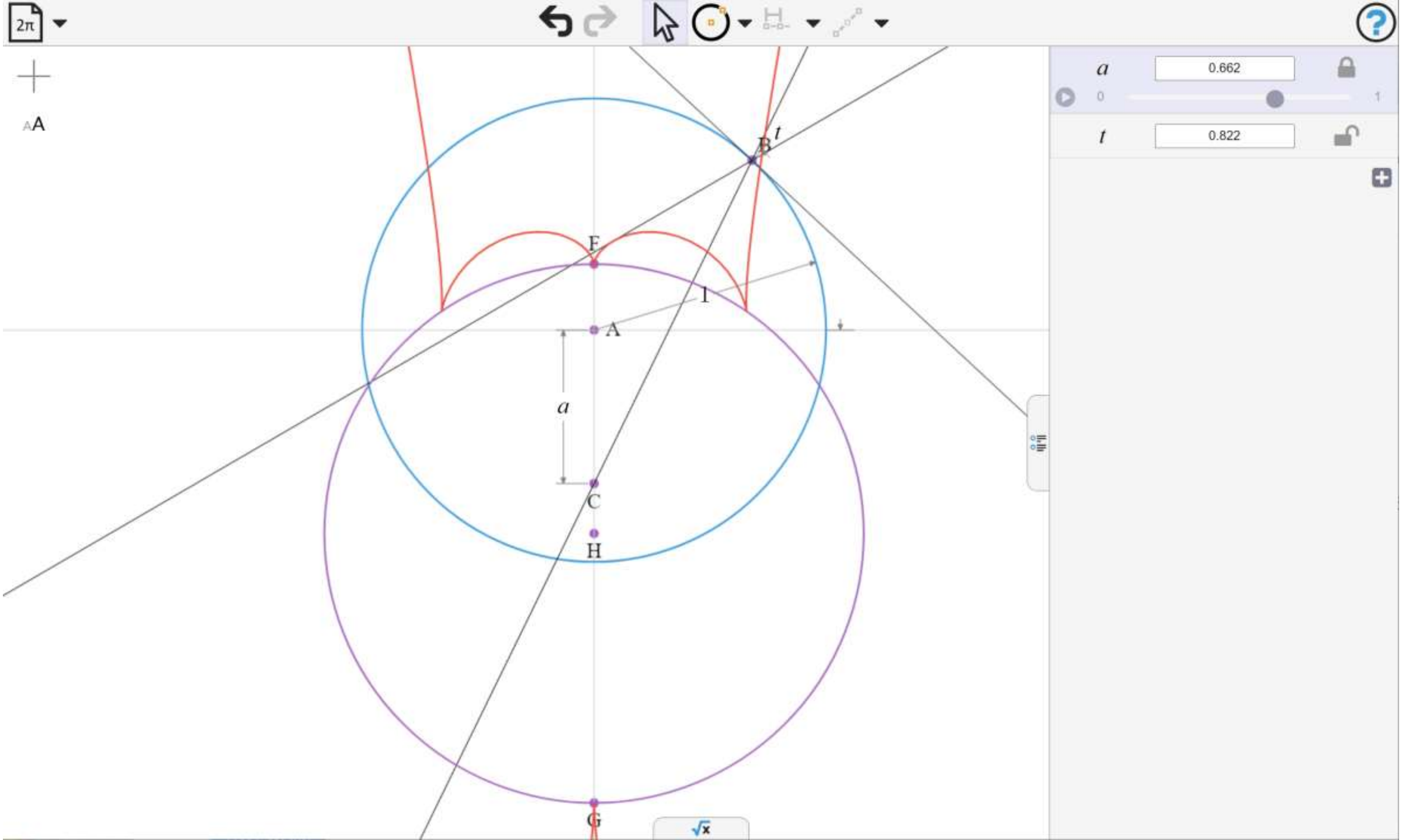


Observe that the on-axis cusps appear to lie on this circle.

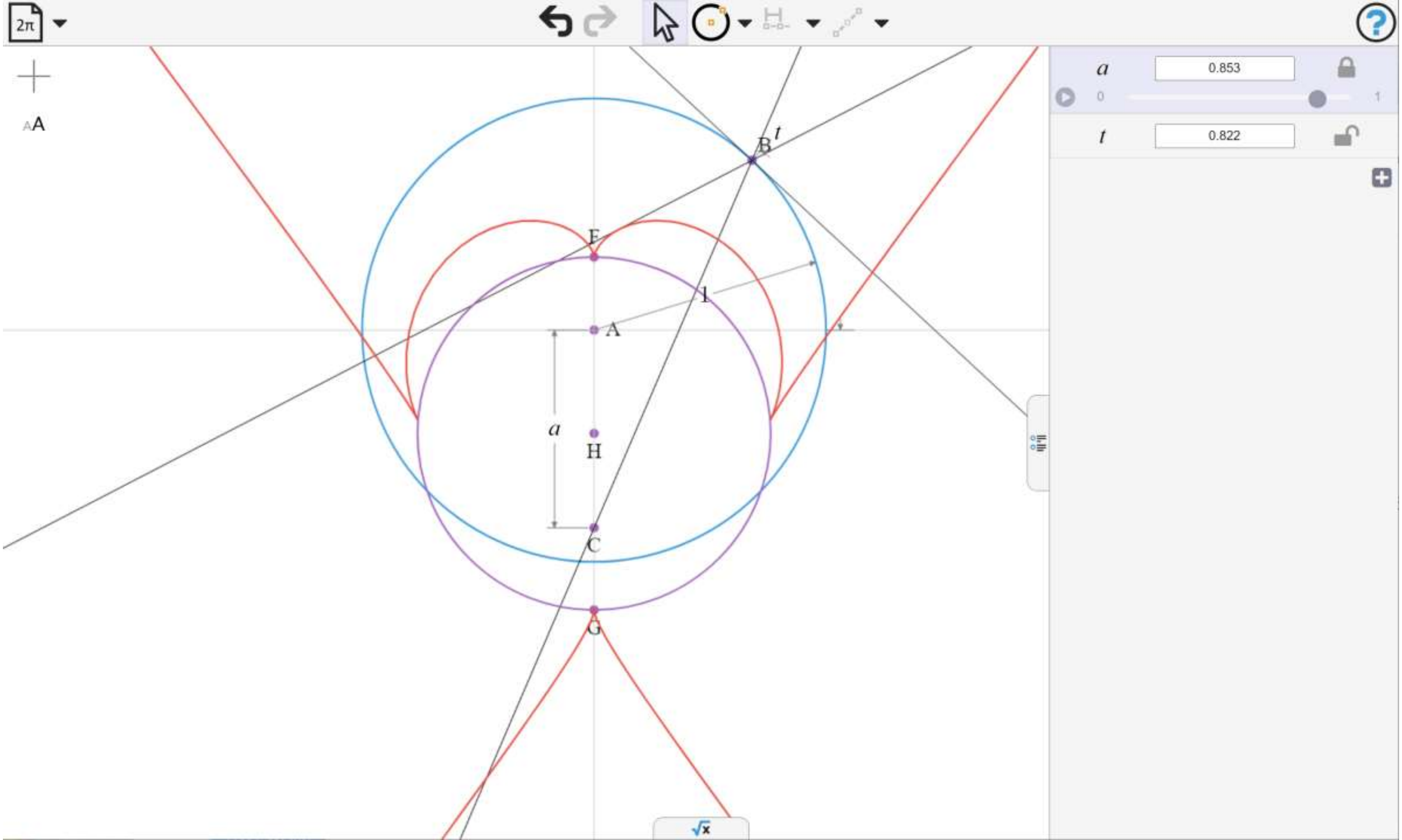

2π
AA
H
B
t
F
1
A
a
C
a
0.366
0
1
t
0.822



2π
a
0.466
0
1
t
0.822
B
t
F
1
A
a
C
√x



2π
a
0.662
0
1
t
0.822
AA
t
B
F
I
A
a
C
H
G

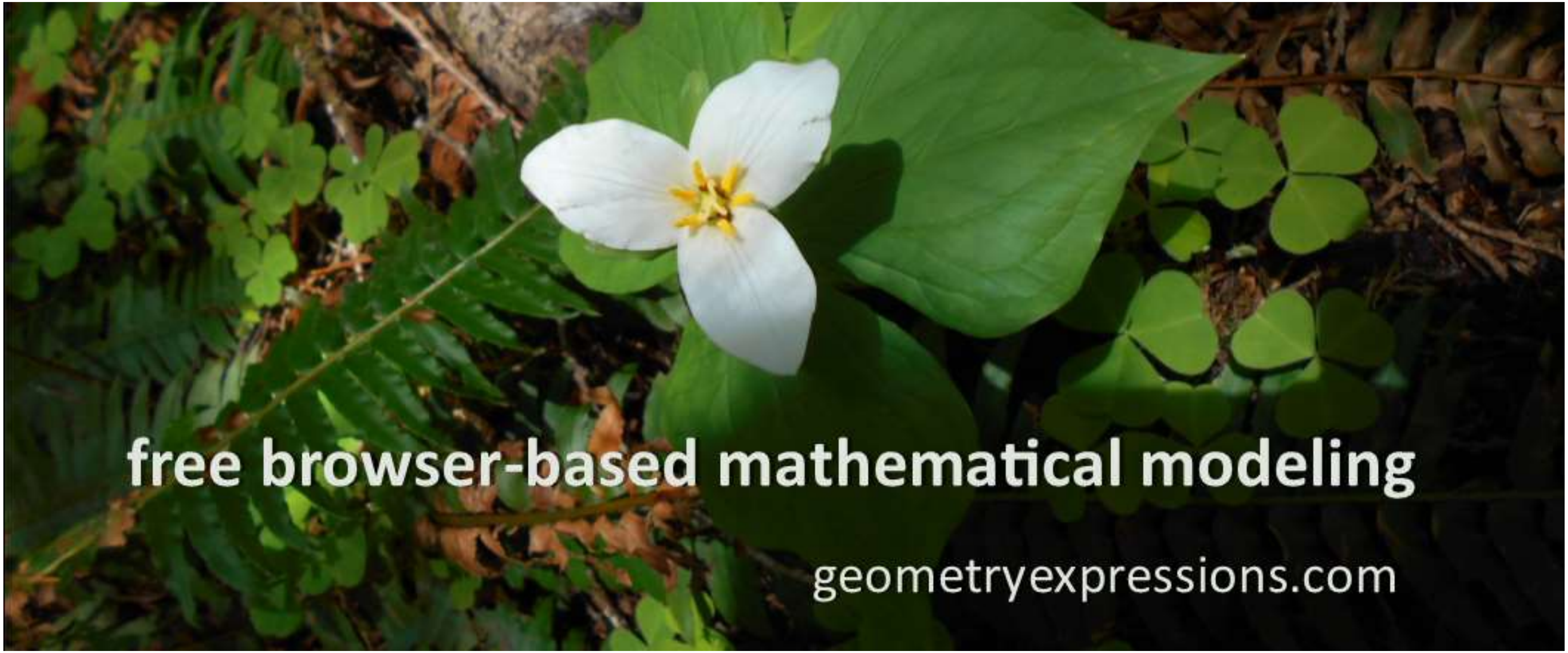


2π
a
0.853
0
1
t
0.822
B
t
E
I
A
a
H
C
G

Take-away questions

Wat is the parametric location of the off-axis cusps?

Does the point with this parameter value in fact lie on the circle whose diameter is the on-axis cusps?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