Automated Generation of Illustrations for Synthetic Geometry Proofs

Predrag Janičić
University of Belgrade, Serbia

Julien Narboux
University of Strasbourg, France

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- In mathematics, especially geometry, illustrations are often very valuable, but almost always just an informal content
- Links between proofs and illustrations are loose
- However, proofs, in some cases, can carry information for illustrations

Existing approaches

- Visualization of statements:
  - using algebraic methods and computations (Gao, Wang)
  - within dynamic geometry tools (GeoGebra etc)
- Visualization of proofs:
  - Full angle method (Wilson and Fleuriot)
  - JGEX algebraic methods (Ye et. al.)
  - PCoq : heuristic for constraint solving
  - Some of the above do not support introducing new points
- In all approaches: visualization rules are hard-coded

Background: Coherent Logic (CL)

- A FOL formula is said to be *coherent* if it is of the form:

$$A_1(\vec{x}) \wedge \ldots \wedge A_n(\vec{x}) \Rightarrow \exists \vec{y}(B_1(\vec{x},\vec{y}) \vee \ldots \vee B_m(\vec{x},\vec{y}))$$

  where universal closure is assumed, A_i denote atomic formulae, B_i denote conjunctions of atomic formulae
- CL is simple, allows simple forward chaining proofs
- Human-readable, natural language proofs but also machine verifiable proofs can be easily obtained
- Any first-order theory can be translated into CL
- Several automated theorem provers for CL, one recent: Larus (Janičić/Narboux)[1]

[1] https://github.com/janicicpredrag/Larus/

Coherent Logic: Toy Example

Consider the following set of axioms:
ax1: $\forall x\ (p(x) \Rightarrow r(x) \vee q(x))$
ax2: $\forall x\ (q(x) \Rightarrow \bot)$
and the following conjecture that can be proved as a CL theorem:
$\forall x\ (p(x) \Rightarrow r(x))$

Consider arbitrary a such that: $p(a)$. It should be proved that $r(a)$.

1. $r(a) \vee q(a)$ (by MP, from $p(a)$ using axiom ax1; instantiation: $X \mapsto a$)
2. Case $r(a)$:
3. Proved by assumption! (by QEDas)
4. Case $q(a)$:
5. $\bot$ (by MP, from $q(a)$ using axiom ax2; instantiation: $X \mapsto a$)
6. Contradiction! (by QEDefq)
7. Proved by case split! (by QEDcs, by $r(a), q(a)$)

Basic Idea for Generating Illustrations

- If we know how to visually interpret proof steps that introduce new objects or facts, we can produce a complete illustration
- The illustration is based on a sequence of such objects in one universum (i.e., model)
- A natural choice for the universum is Cartesian space
- This idea is well-suited to CL and to forward chaining proofs

Rule Applications

- Consider the axiom:

$$\forall x, y\ (point(x) \wedge point(y) \Rightarrow \exists z\ between(x, z, y))$$

- It may have attached the visual interpretation:
  ,,for two Cartesian points *a* and *b*, a Cartesian point *c* is created as the midpoint of *ab*"
- If the axiom is applied to the points *a* and *b*, with associated Cartesian coordinates $(2, 5)$ and $(4, 11)$, then the new witness point will have the associated Cartesian coordinates $(3, 8)$
- Not only new witnesses can be created, but also some new features can also be illustrated

Illustrating Proof Branches

- It makes no much sense to illustrate all proof branches
- An illustration is created for at most one proof branch:
  - If all proof branches end with contradiction, then they all belong to some upper contradictory proof branch, and we illustrate neither of them.
  - If there are some proof branches that do not end with contradiction, then one that corresponds to the model being built should be illustrated.

Premises and Initial Configuration

- How do we start the illustration in the first place?
- We prove theorems of the form:
  $A_0(\vec{x}) \wedge \ldots \wedge A_{n-1}(\vec{x}) \Rightarrow \exists \vec{y}(B_0(\vec{x}, \vec{y}) \vee \ldots \vee \ B_{m-1}(\vec{x}, \vec{y}))$
- In order to build the initial illustration we need some constants $\vec{a}$ such that: $A_0(\vec{a}) \wedge \ldots \wedge A_{n-1}(\vec{a})$ holds.
- How can we find and illustrate such objects?
- By proving and illustrating the conjecture
  $\exists \vec{x}(A_0(\vec{x}) \wedge \ldots \wedge A_{n-1}(\vec{x}))$.
- If the above conjecture is not theorem, the premises are inconsistent and the statement is trivially valid

Illustrations for Axioms and Theorems

- The visual interpretation of each theorem can either be
  - provided by the user or
  - produced automatically recursively, using the same approach.
- Ultimately, what we need are only visual interpretations of all axioms, provided by a human.

Randomization

- In order to make illustrations partly unpredictable and more interesting, some randomization may be added to the visual interpretations

Implementation

- We implemented the described method within our automated theorem prover for coherent logic, Larus
- For the target language we chose the GCL language – a rich, special purpose language for mathematical, especially geometry illustrations.
- For each axiom, the user has to provide a corresponding visualisation in terms of a GCLC function; for example:

For any two points A and B, there is a point C such that $bet(A, B, C)$	`random r`
	`expression r' {1+r}`
	`towards C A B r'`

- Step-by-step visuelization (animations) supported

Example Theorem (from *Elements*)

Theorem

proposition_11 : $\forall A\ \forall B\ \forall C\ (betS(A, C, B) \Rightarrow \exists X\ (per(A, C, X)))$

Proof:

Consider arbitrary a, b, c such that: $betS(a, c, b)$. It should be proved that $\exists X\ \ per(a, c, X)$.

1. Let w be such that $betS(a, c, w) \wedge cong(c, w, a, c)$ (by MP, from $betS(a, c, b)$, $betS(a, c, b)$ using axiom lemma_extension)
2. Let $w1$ be such that $equilateral(a, w, w1) \wedge triangle(a, w, w1)$ (by MP, from $betS(a, c, w) \wedge cong(c, w, a, c)$ using axiom proposition_01)
3. $w1 = c \vee w1 \neq c$ (by MP, using axiom eq_excluded_middle)
4. Case $w1 = c$:
5. $col(a, w, w1)$ (by MP, from $betS(a, c, w) \wedge cong(c, w, a, c)$, $w1 = c$ using axiom colEqSub2)
6. $\perp$ (by MP, from $col(a, w, w1)$, $equilateral(a, w, w1) \wedge triangle(a, w, w1)$ using axiom nnncolNegElim)
7. Contradiction! (by QEDefq)
8. Case $w1 \neq c$:
9. $per(a, c, w1)$ (by MP, from $betS(a, c, w) \wedge cong(c, w, a, c)$, $betS(a, c, w) \wedge cong(c, w, a, c)$, $equilateral(a, w, w1) \wedge triangle(a, w, w1)$, $w1 \neq c$ using axiom defrightangle2)
10. Proved by assumption! (by QEDas)
11. Proved by case split! (by QEDcs, by $w1 = c$, $w1 \neq c$)

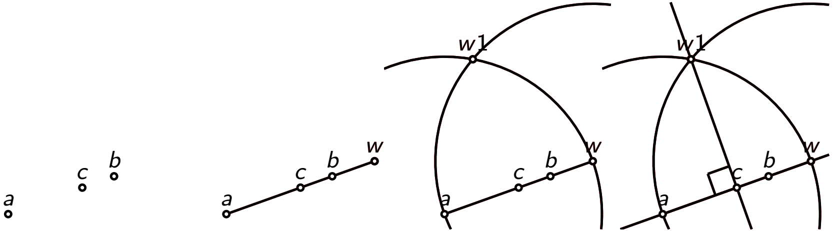
A code provided by the user:

```
% fof(lemma_extension,axiom, (! [A,B,P,Q] : (? [X] :
     ((( A != B ) & ( P != Q )) => ((betS(A,B,X) & cong(B,X,P,Q))))))).
procedure lemma_extension { A B P Q X } {
  distance d1 A B
  distance d2 P Q
  expression r { 1+(d1/d2) }
  towards X A B r
  drawsegment A X
  cmark X
}
```

Example

A code generated by the prover:

```
% ----- Proof illustration -----
include lemma_extension.gcl
include proposition_01.gcl
include defrightangle2.gcl
include proposition_11_exists.gcl
%------------------------------
procedure proposition_11 { a b c w  }  {
  call lemma_extension { a c a c w }
  mark_t w
  call proposition_01 { a w w1 }
  mark_t w1
  % --- Illustration for branch 2
  call defrightangle2 { a c w1 w }
}
%------------------------------
call proposition_11_exists { a b c }
call proposition_11 { a b c w  }
```



Four steps in illustration of the proposition 11

Conclusions and Future Work

- The approach is a sort of constraint solving based on theorem proving
- Proofs of existence depend on the axioms provided (for instance, an axiom that enables angle trisection)
- The approach is:

simple	as it is a small extension to a CL prover
modular	as all illustrations rely only on visual interpretations of axioms used
flexible	as one can provide different visual counterparts of the axioms, but also of particular lemmas used