Realizations of Rigid Graphs

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(joint work with Jose Capco, Matteo Gallet, Georg Grasegger,
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Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences

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RISC Hagenberg

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Rigid and Non-Rigid Graphs

Notation: Let $G = (V, E)$ be a graph, and let $\lambda\colon E \to \mathbb{R}_{>0}$ be a **labeling** of its edges, that is **realizable** (as lengths in $\mathbb{R}^2$).

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Example: $G = (V, E)$ with

$$V = \{1, 2, 3, 4\},$$
$$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\}$$

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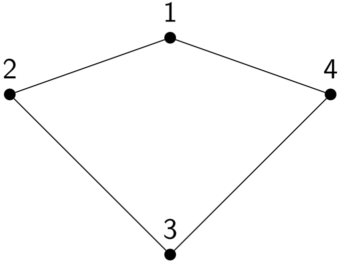

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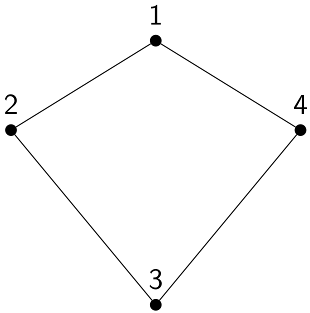

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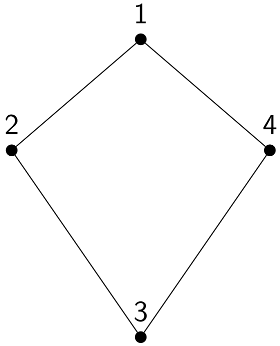

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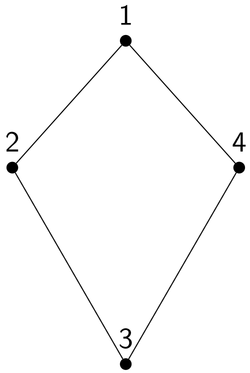

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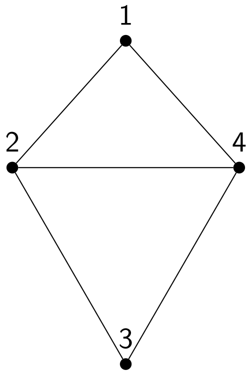

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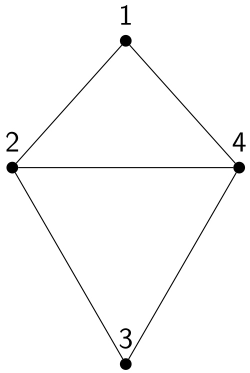

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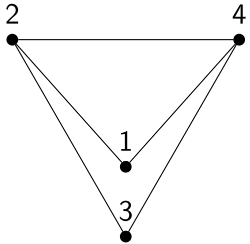

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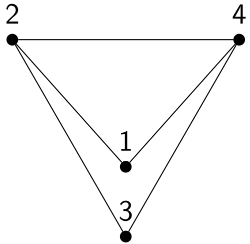

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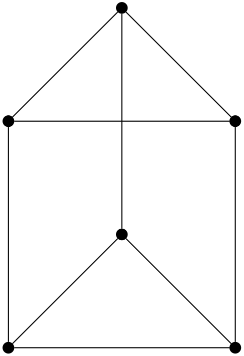
Three-Prism Graph

Is this graph rigid?



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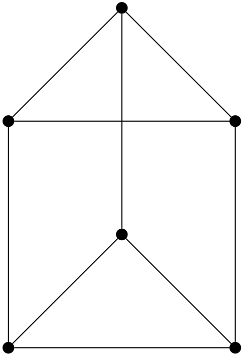

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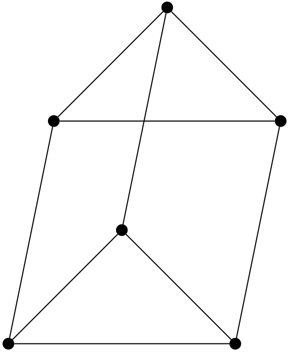

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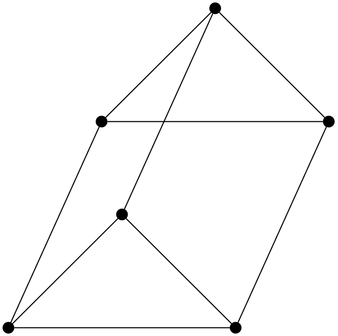

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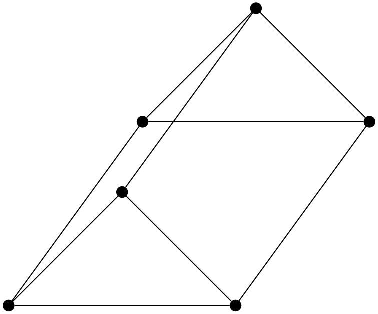

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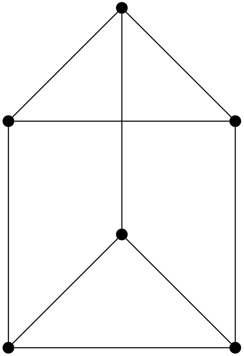

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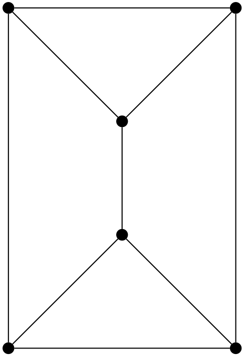

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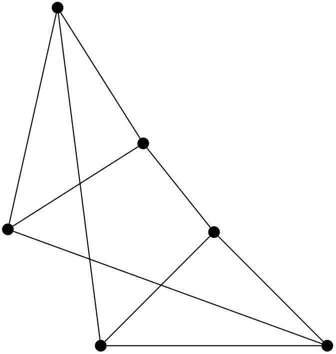

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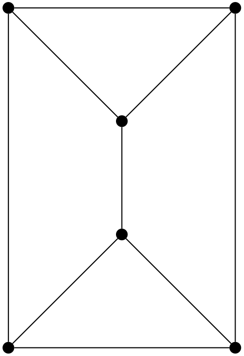

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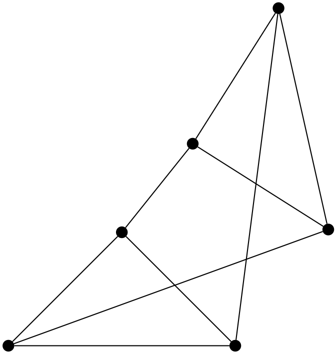

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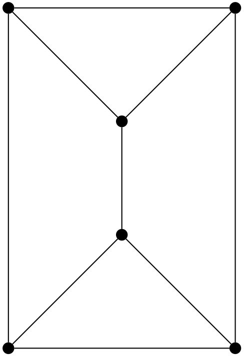

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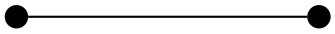
Theorem. (Geiringer 1927, Laman 1970)
A graph $G = (V, E)$ is minimally rigid if and only if

1. $|E| = 2|V| - 3$,
2. $|E'| \leqslant 2|V'| - 3$ for each subgraph $G' = (V', E')$ of G.

Some Minimally Rigid Graphs

All minimally rigid graphs with $2 \leqslant n \leqslant 5$ vertices:

$n = 2$: 

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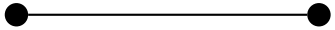

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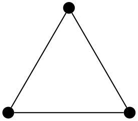

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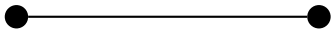

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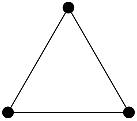

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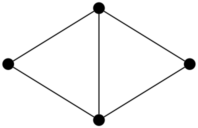

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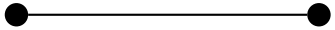

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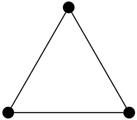

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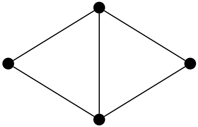

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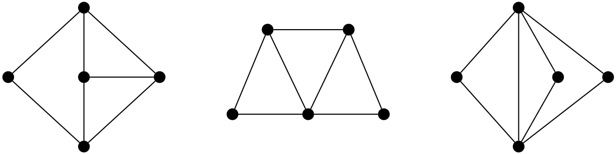

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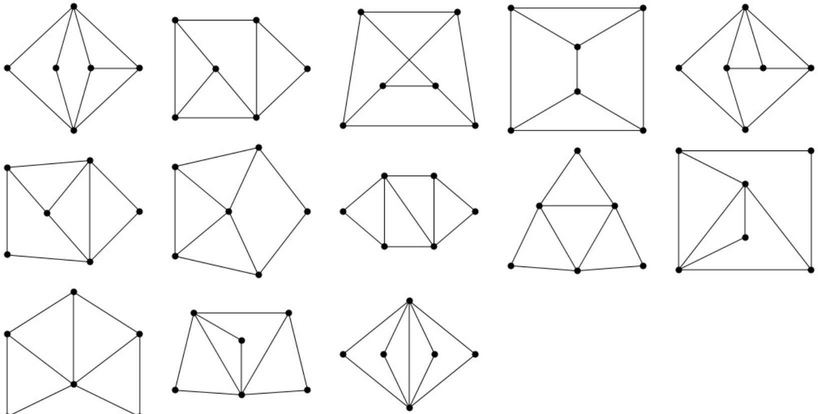
All minimally rigid graphs with 6 vertices:



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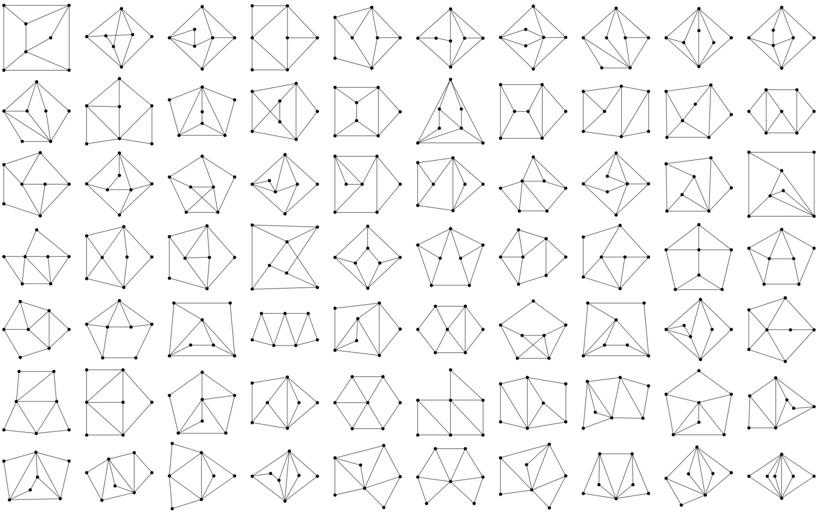
There are 70 minimally rigid graphs with 7 vertices:



Enumeration of Minimally Rigid Graphs

Number of minimally rigid graphs with n vertices:

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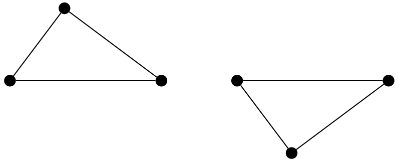

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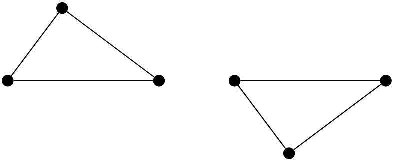

Minimally rigid graph with 4 vertices: ?

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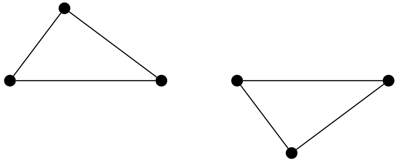
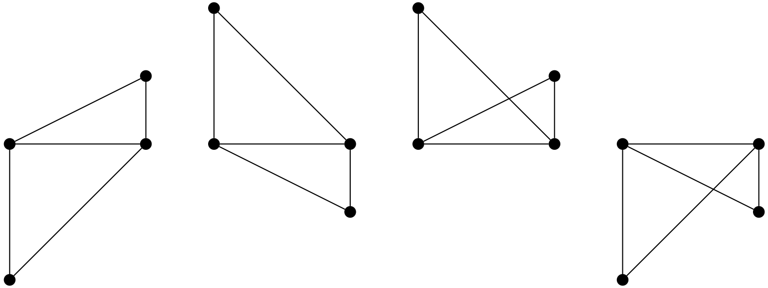


Minimally rigid graph with 4 vertices: 4 realizations



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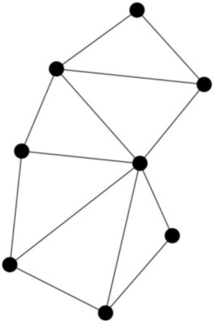

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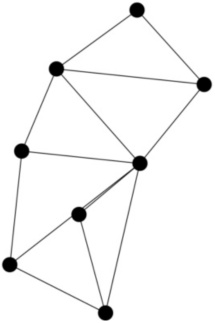

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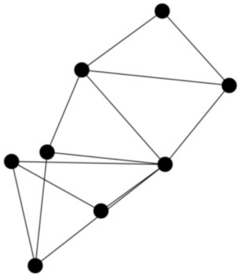

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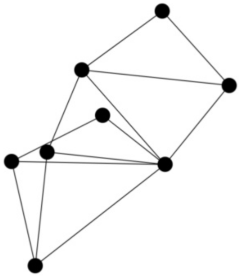

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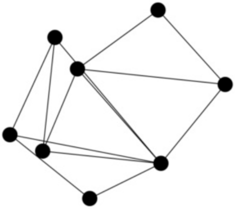

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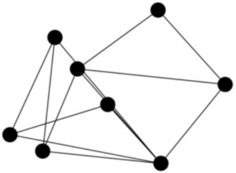

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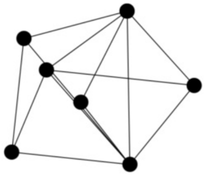

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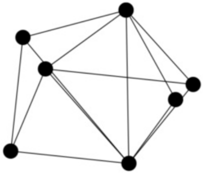

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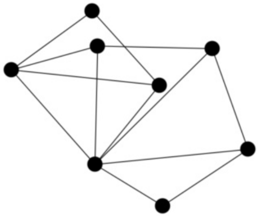

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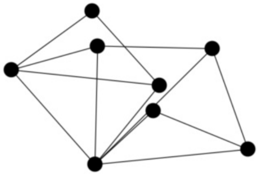

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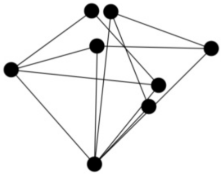

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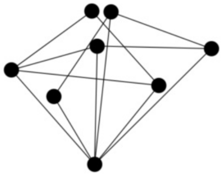

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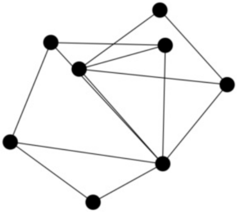

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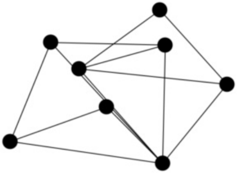

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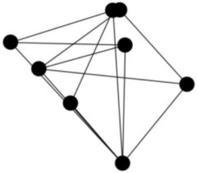

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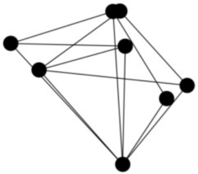

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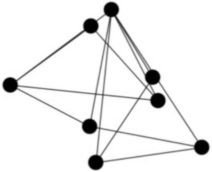

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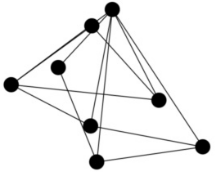

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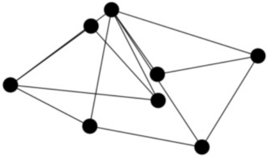

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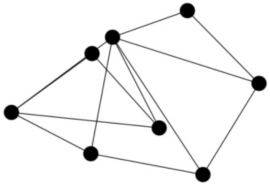

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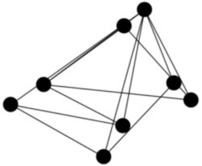

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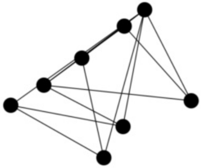

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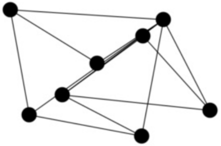

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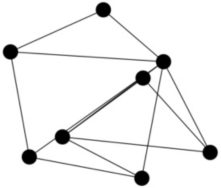

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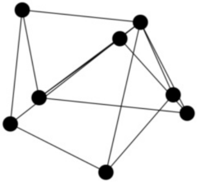

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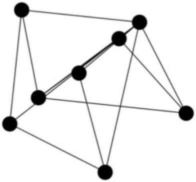

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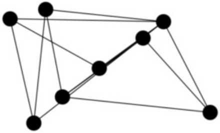

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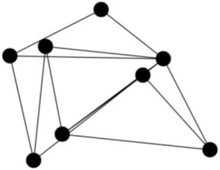

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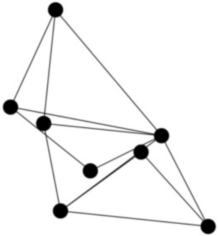

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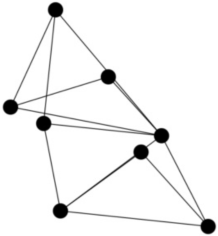

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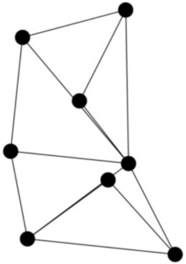

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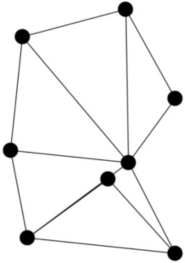

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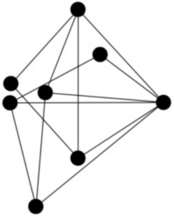

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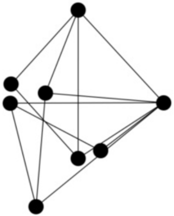

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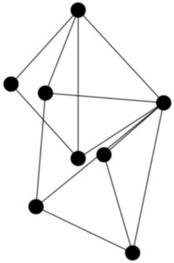

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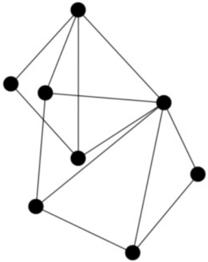

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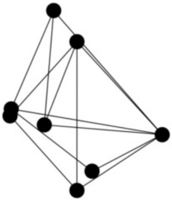

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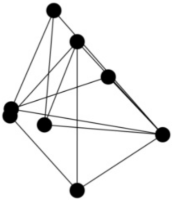

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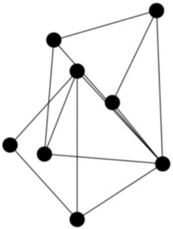

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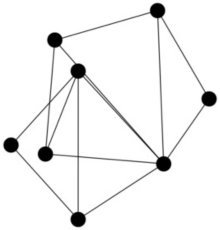

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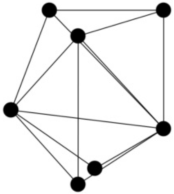

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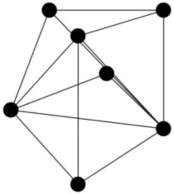

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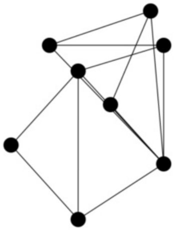

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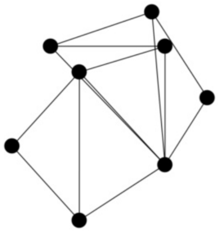

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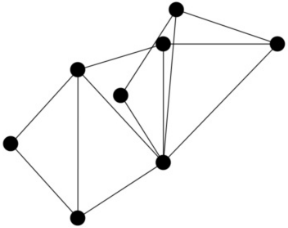

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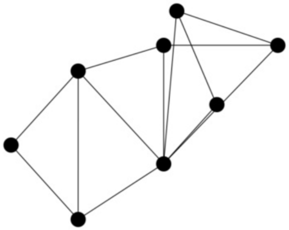

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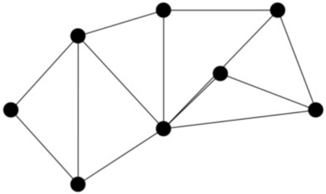

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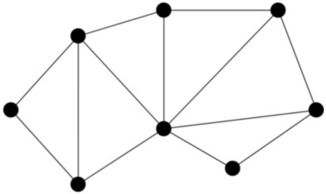

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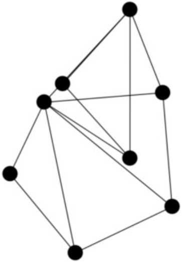

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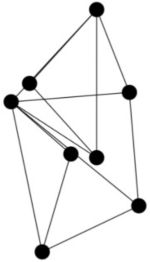

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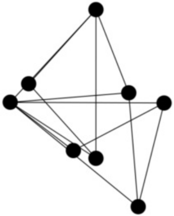

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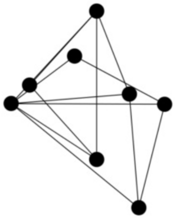

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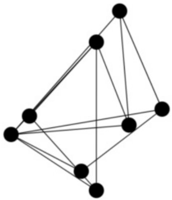

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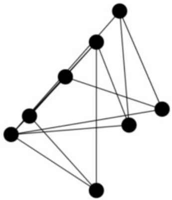

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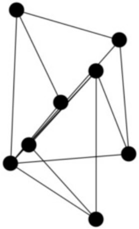

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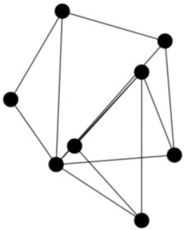

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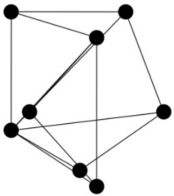

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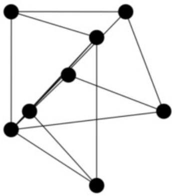

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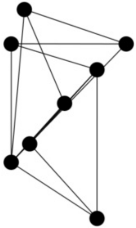

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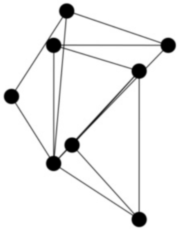

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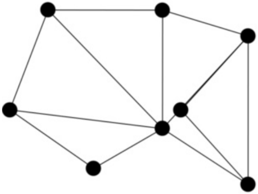

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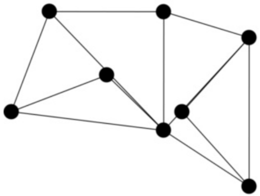

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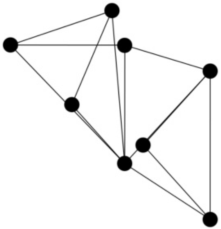

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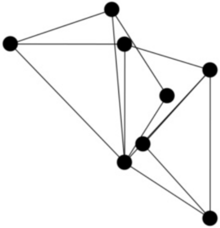
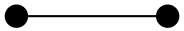


Realizations of H1 Graphs

Definition: An **H1 graph** is a minimally rigid graph that can be obtained by successively connecting a new vertex with two existing ones, starting with the graph 

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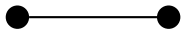
Definition: An **H1 graph** is a minimally rigid graph that can be obtained by successively connecting a new vertex with two existing ones, starting with the graph 

Number of realizations:

- Let $G = (V, E)$ be an H1 graph.
- Fix a realizable labeling $\lambda\colon E \to \mathbb{R}_{>0}$.
- Fix the positions of the first two vertices, respecting $\lambda(1, 2)$.
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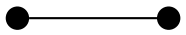
Definition: An **H1 graph** is a minimally rigid graph that can be obtained by successively connecting a new vertex with two existing ones, starting with the graph 

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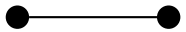
Definition: An **H1 graph** is a minimally rigid graph that can be obtained by successively connecting a new vertex with two existing ones, starting with the graph 

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Definition: The **Laman number** $\mathrm{Lam}(G)$ of a minimally rigid graph G is the number of realizations of G, for a generic realizable labeling λ.

Minimally Rigid Graphs that are not H1

Question: What about minimally rigid graphs that are not H1?

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Set up a system of equations:

- Let (x_v, y_v) be the coordinates of vertex v.
- For $\{u, v\} \in E$:

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(u, v)^2.$$

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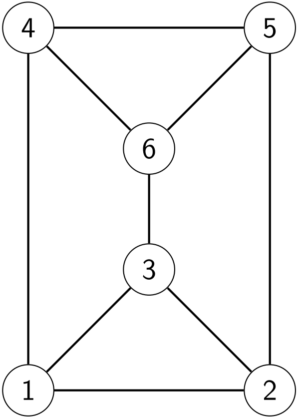
Convention: From now on we work over the complex numbers:

- $\lambda \colon E \to \mathbb{C}$
- $(x_v, y_v) \in \mathbb{C}^2$

Example: Three-Prism Graph



$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = \lambda(1,2)^2$$
$$(x_1 - x_3)^2 + (y_1 - y_3)^2 = \lambda(1,3)^2$$
$$(x_1 - x_4)^2 + (y_1 - y_4)^2 = \lambda(1,4)^2$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = \lambda(2,3)^2$$
$$(x_2 - x_5)^2 + (y_2 - y_5)^2 = \lambda(2,5)^2$$
$$(x_3 - x_6)^2 + (y_3 - y_6)^2 = \lambda(3,6)^2$$
$$(x_4 - x_5)^2 + (y_4 - y_5)^2 = \lambda(4,5)^2$$
$$(x_4 - x_6)^2 + (y_4 - y_6)^2 = \lambda(4,6)^2$$
$$(x_5 - x_6)^2 + (y_5 - y_6)^2 = \lambda(5,6)^2$$

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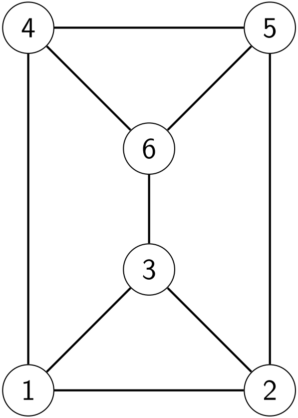

$$
\begin{aligned}
(\quad x_2)^2 + (\quad y_2)^2 &= \lambda(1,2)^2 \\
(\quad x_3)^2 + (\quad y_3)^2 &= \lambda(1,3)^2 \\
(\quad x_4)^2 + (\quad y_4)^2 &= \lambda(1,4)^2 \\
(x_2 - x_3)^2 + (y_2 - y_3)^2 &= \lambda(2,3)^2 \\
(x_2 - x_5)^2 + (y_2 - y_5)^2 &= \lambda(2,5)^2 \\
(x_3 - x_6)^2 + (y_3 - y_6)^2 &= \lambda(3,6)^2 \\
(x_4 - x_5)^2 + (y_4 - y_5)^2 &= \lambda(4,5)^2 \\
(x_4 - x_6)^2 + (y_4 - y_6)^2 &= \lambda(4,6)^2 \\
(x_5 - x_6)^2 + (y_5 - y_6)^2 &= \lambda(5,6)^2
\end{aligned}
$$

- Take care of translations: $(x_1, y_1) = (0, 0)$

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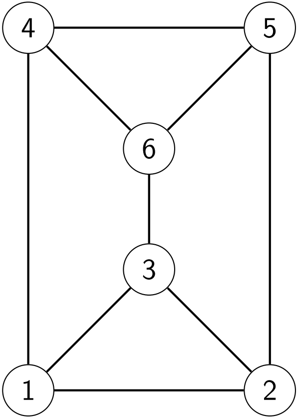

$$
\begin{aligned}
y_2 &= \lambda(1,2) \\
(\phantom{x_3 -{}} x_3)^2 + (\phantom{y_2 -{}} y_3)^2 &= \lambda(1,3)^2 \\
(\phantom{x_3 -{}} x_4)^2 + (\phantom{y_2 -{}} y_4)^2 &= \lambda(1,4)^2 \\
(\phantom{x_3 -{}} x_3)^2 + (y_2 - y_3)^2 &= \lambda(2,3)^2 \\
(\phantom{x_3 -{}} x_5)^2 + (y_2 - y_5)^2 &= \lambda(2,5)^2 \\
(x_3 - x_6)^2 + (y_3 - y_6)^2 &= \lambda(3,6)^2 \\
(x_4 - x_5)^2 + (y_4 - y_5)^2 &= \lambda(4,5)^2 \\
(x_4 - x_6)^2 + (y_4 - y_6)^2 &= \lambda(4,6)^2 \\
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\end{aligned}
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- Take care of translations: $(x_1, y_1) = (0, 0)$
- Take care of rotations: $x_2 = 0$ and $y_2 > 0$

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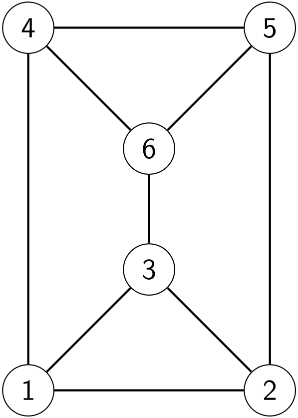

$$
\begin{aligned}
y_2 &= \lambda(1,2) \\
(x_3)^2 + (y_3)^2 &= \lambda(1,3)^2 \\
(x_4)^2 + (y_4)^2 &= \lambda(1,4)^2 \\
(x_3)^2 + (y_2 - y_3)^2 &= \lambda(2,3)^2 \\
(x_5)^2 + (y_2 - y_5)^2 &= \lambda(2,5)^2 \\
(x_3 - x_6)^2 + (y_3 - y_6)^2 &= \lambda(3,6)^2 \\
(x_4 - x_5)^2 + (y_4 - y_5)^2 &= \lambda(4,5)^2 \\
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\end{aligned}
$$

- Take care of translations: $(x_1, y_1) = (0, 0)$
- Take care of rotations: $x_2 = 0$ and $y_2 > 0$

Question: How many solutions does this system have?

Gröbner Basis Approach

- Not feasible for symbolic parameters $\lambda(i, j)$

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- Not feasible for symbolic parameters $\lambda(i, j)$
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- Do the computation modulo $p = 2^{31} - 1$:

$$\{y_3 + 1727076644,\ x_5 x_6 + 1073741823 x_6^2 + y_5 y_6 + 1073741823 y_6^2 + 2147483458 y_5 + 1073746572,\ x_4 x_6 + 1073741823 x_6^2 + y_4 y_6 + 1073741823 y_6^2 + 2147472199,$$
$$x_3 x_6 + 1073741823 x_6^2 + 1073741823 y_6^2 + 420407003 y_6 + 2147476519,\ x_3 y_5 + 1449935236 x_4 y_5 + 87139559 x_5 y_5 + 821582392 y_4 x_6 +$$
$$534432936 y_5 x_6 + 2127003394 x_3 y_6 + 393122455 x_4 y_6 + 739525427 x_5 y_6 + 1428199694 x_6 y_6 + 1318362776 x_3 + 45332622 x_4 + 1666067743 x_5 + 1402190174 x_6,$$
$$x_5^2 + y_5^2 + 2147483269 y_5 + 2147482119,\ y_4 x_5 + 1431835485 x_4 y_5 + 1585512332 x_5 y_5 + 2099455504 y_4 x_6 + 1274481640 y_5 x_6 + 1926461619 x_3 y_6 + 1819204411 x_4 y_6 +$$
$$2064309228 x_5 y_6 + 1860755017 x_6 y_6 + 758303990 x_3 + 504327305 x_4 + 513732789 x_5 + 1018326077 x_6,\ x_4 x_5 + y_4 y_5 + 2147483458 y_5 + 2147472715,$$
$$y_4^2 + 544418756 y_4 y_5 + 47332294 y_5^2 + 1603064889 y_4 y_6 + 1508400303 y_5 y_6 + 591751051 y_6^2 + 1072510925,\ x_4 y_4 + 1252848948 x_4 y_5 + 1309508129 x_5 y_5 + 2016071435 y_4 x_6 +$$
$$1654953235 y_5 x_6 + 1839606594 x_3 y_6 + 577627465 x_4 y_6 + 876148120 x_5 y_6 + 335588542 x_6 y_6 + 2136682920 x_3 + 1038483051 x_4 + 157778557 x_5 + 540431639 x_6,$$
$$x_3 y_4 + 204011627 x_4 y_5 + 839002279 x_5 y_5 + 368180718 y_4 x_6 + 1641249205 y_5 x_6 + 430135887 x_3 y_6 + 486556477 x_4 y_6 + 1706891994 x_5 y_6 + 83415671 x_6 y_6 + 123469149 x_3 +$$
$$554422930 x_4 + 1257780688 x_5 + 1936702634 x_6,\ x_4^2 + 1603064891 y_4 y_5 + 2100151353 y_5^2 + 544418758 y_4 y_6 + 639083344 y_5 y_6 + 1555732596 y_6^2 + 1074934697,$$
$$x_3^2 + 1527353090,\ y_6^3 + 72446234 x_3 x_4 + 191839850 x_3 x_5 + 1293615843 y_4 y_5 + 2115905836 y_5^2 + 158590087 x_6^2 + 808924606 y_4 y_6 + 945043470 y_5 y_6 + 57464572 y_6^2 +$$
$$1051760435 y_4 + 458639039 y_5 + 890226333 y_6 + 306458357,\ x_6 y_6^2 + 1202942319 x_4 y_5 + 891621123 x_5 y_5 + 694981073 y_4 x_6 + 1268149853 y_5 x_6 +$$
$$566843284 x_3 y_6 + 1579449712 x_4 y_6 + 2096672325 x_5 y_6 + 217935702 x_6 y_6 + 1838771945 x_3 + 1574100689 x_4 + 890711649 x_5 + 527754025 x_6,$$
$$y_5 y_6^2 + 1397298562 x_3 x_4 + 1093626759 x_3 x_5 + 1874498615 y_4 y_5 + 410806791 y_5^2 + 34715881 x_6^2 + 1602680419 y_4 y_6 + 1365806073 y_5 y_6 + 1574368257 y_6^2 +$$
$$1986672592 y_4 + 1454700418 y_5 + 207782012 y_6 + 817238271,\ x_5 y_6^2 + 906551028 x_4 y_5 + 2088326233 x_5 y_5 + 983660499 y_4 x_6 + 2020744231 y_5 x_6 + 438982960 x_3 y_6 +$$
$$105460105 x_4 y_6 + 1791795415 x_5 y_6 + 752681903 x_6 y_6 + 1243232341 x_3 + 236567207 x_4 + 2039336095 x_5 + 204724127 x_6,\ y_4 y_6^2 + 1798033564 x_3 x_4 + 1368970181 x_3 x_5 +$$
$$2111288438 y_4 y_5 + 2116525889 y_5^2 + 631579871 x_6^2 + 2098374939 y_4 y_6 + 14559548 y_5 y_6 + 265925976 y_6^2 + 768097244 y_4 + 197849421 y_5 + 1272087803 y_6 + 1950925264,$$
$$x_4 y_6^2 + 2000108329 x_4 y_5 + 138882411 x_5 y_5 + 1964621882 y_4 x_6 + 1562649152 y_5 x_6 + 274800980 x_3 y_6 + 381168929 x_4 y_6 + 1561080504 x_5 y_6 + 646135501 x_6 y_6 +$$
$$1252024999 x_3 + 1828948462 x_4 + 1907059409 x_5 + 1062878925 x_6,\ x_3 y_6^2 + 1940064434 x_4 y_5 + 1699323466 x_5 y_5 + 2767389 y_4 x_6 + 309430653 y_5 x_6 +$$
$$1746152111 x_3 y_6 + 1486922955 x_4 y_6 + 1042873400 x_5 y_6 + 1877302158 x_6 y_6 + 898857598 x_3 + 2023749908 x_4 + 1369459334 x_5 + 1937240806 x_6,$$
$$x_6^2 y_6 + 1859350309 x_3 x_4 + 828165967 x_3 x_5 + 1319416915 y_4 y_5 + 1281531769 y_5^2 + 416445396 x_6^2 + 555896977 y_4 y_6 + 838162654 y_5 y_6 + 1094699319 y_6^2 +$$
$$1025635396 y_4 + 758820774 y_5 + 1932663106 y_6 + 902372666,\ y_5 x_6 y_6 + 1776737250 x_4 y_5 + 1335234339 x_5 y_5 + 197659465 y_4 x_6 + 388691694 y_5 x_6 +$$
$$1214819713 x_3 y_6 + 1236939013 x_4 y_6 + 1895585096 x_5 y_6 + 1457663787 x_6 y_6 + 1908824636 x_3 + 1937866443 x_4 + 906541898 x_5 + 1779256072 x_6,$$
$$y_4 x_6 y_6 + 392800087 x_4 y_5 + 43314235 x_5 y_5 + 1752015765 y_4 x_6 + 697637736 y_5 x_6 + 1174862040 x_3 y_6 + 1726470482 x_4 y_6 + 524280549 x_5 y_6 + 1783594194 x_6 y_6 +$$
$$777027038 x_3 + 1196924612 x_4 + 669351278 x_5 + 136564514 x_6,\ y_5^2 y_6 + 769157270 x_3 x_4 + 30129177 x_3 x_5 + 147541859 y_4 y_5 + 696342885 y_5^2 + 953052903 x_6^2 + 63094058 y_4 y_6 +$$
$$1607776536 y_5 y_6 + 2003959420 y_6^2 + 1657122998 y_4 + 1041341194 y_5 + 643382090 y_6 + 298205040,\ x_5 y_5 y_6 + 1361368571 x_4 y_5 + 443005480 x_5 y_5 + 749246637 y_4 x_6 +$$
$$556781711 y_5 x_6 + 268588588 x_3 y_6 + 179323388 x_4 y_6 + 260672145 x_5 y_6 + 542764427 x_6 y_6 + 2031844241 x_3 + 112806238 x_4 + 2024966158 x_5 + 1634398898 x_6,$$
$$y_4 y_5 y_6 + 1495723262 x_3 x_4 + 1150552515 x_3 x_5 + 647627904 y_4 y_5 + 834052394 y_5^2 + 680400990 x_6^2 + 703082161 y_4 y_6 + 1261907640 y_5 y_6 + 1146980666 y_6^2 +$$
$$339024153 y_4 + 1829077048 y_5 + 1120614065 y_6 + 420646718,\ x_4 y_5 y_6 + 391789202 x_4 y_5 + 1778622432 x_5 y_5 + 32574434 y_4 x_6 + 638884222 y_5 x_6 +$$
$$2008976092 x_3 y_6 + 1158838637 x_4 y_6 + 298082231 x_5 y_6 + 579017100 x_6 y_6 + 541015847 x_3 + 1347513279 x_4 + 1774560872 x_5 + 1614705109 x_6,$$
$$x_3 x_5 y_6 + 1581681716 x_3 x_4 + 486946881 x_3 x_5 + 421622009 y_4 y_5 + 1075313850 y_5^2 + 1564800523 x_6^2 + 198951616 y_4 y_6 + 1466002977 y_5 y_6 + 932669036 y_6^2 + 248319512 y_4 +$$
$$862020011 y_5 + 649537600 y_6 + 815933435,\ x_3 x_4 y_6 + 1343545648 x_3 x_4 + 1023324514 x_3 x_5 + 40371239 y_4 y_5 + 1905289341 y_5^2 + 1639954889 x_6^2 + 786545101 y_4 y_6 +$$
$$1219192433 y_5 y_6 + 321512152 y_6^2 + 1631897898 y_4 + 850776521 y_5 + 530499711 y_6 + 2036743747,\ x_6^3 + 1359379754 x_4 y_5 + 4699239 x_5 y_5 + 1446967796 y_4 x_6 +$$
$$260472488 y_5 x_6 + 701675423 x_3 y_6 + 1889155319 x_4 y_6 + 112548169 x_5 y_6 + 1629096917 x_6 y_6 + 658508665 x_3 + 820850436 x_4 + 665336977 x_5 + 1707190979 x_6,$$
$$y_5 x_6^2 + 1013046601 x_3 x_4 + 969596453 x_3 x_5 + 1553889292 y_4 y_5 + 1185309841 y_5^2 + 1987921573 x_6^2 + 1033458441 y_4 y_6 + 1320068753 y_5 y_6 + 1102491211 y_6^2 +$$
$$1104911459 y_4 + 1375116864 y_5 + 672833739 y_6 + 626376074,\ y_4 x_6^2 + 2009134087 x_3 x_4 + 1611713763 x_3 x_5 + 168461479 y_4 y_5 + 1706153267 y_5^2 +$$
$$1769015690 x_6^2 + 1182579576 y_4 y_6 + 557864255 y_5 y_6 + 503714053 y_6^2 + 176291393 y_4 + 1354065871 y_5 + 954347026 y_6 + 734410570,$$
$$y_5^2 x_6 + 619010252 x_4 y_5 + 916121455 x_5 y_5 + 1431371638 y_4 x_6 + 969212309 y_5 x_6 + 1949990023 x_3 y_6 + 414782496 x_4 y_6 + 1907745509 x_5 y_6 + 970368126 x_6 y_6 +$$
$$1740320236 x_3 + 1975330810 x_4 + 2143293978 x_5 + 252311982 x_6,\ y_4 y_5 x_6 + 558167487 x_4 y_5 + 433016430 x_5 y_5 + 2075138717 y_4 x_6 + 1434835475 y_5 x_6 +$$
$$531264210 x_3 y_6 + 427467244 x_4 y_6 + 1374860777 x_5 y_6 + 149117380 x_6 y_6 + 1826680361 x_3 + 969629736 x_4 + 766694650 x_5 + 1666548268 x_6,$$
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$$354162717 y_4 + 737861553 y_5 + 816931778 y_6 + 1428529698,\ x_5 y_5^2 + 1136639110 x_4 y_5 + 121108532 x_5 y_5 + 2127022098 y_4 x_6 + 701800649 y_5 x_6 +$$
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$$y_4 y_5^2 + 1382099903 x_3 x_4 + 1674451197 x_3 x_5 + 1964164303 y_4 y_5 + 610824582 y_5^2 + 1726175807 x_6^2 + 1045412838 y_4 y_6 + 1328732288 y_5 y_6 + 1416893499 y_6^2 +$$
$$509989107 y_4 + 356562705 y_5 + 701591991 y_6 + 90791056,\ x_4 y_5^2 + 1125381690 x_4 y_5 + 343309511 x_5 y_5 + 412315532 y_4 x_6 + 392837310 y_5 x_6 +$$
$$1859774430 x_3 y_6 + 1289634195 x_4 y_6 + 511405427 x_5 y_6 + 2104680646 x_6 y_6 + 1304660656 x_3 + 1431387822 x_4 + 2142663821 x_5 + 395031648 x_6\}$$

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$2111288438\, y_4 y_5 + 2116525889\, y_5^2 + 631579871\, x_6^2 + 2098374939\, y_4 y_6 + 14559548\, y_5 y_6 + 265925976\, y_6^2 + 768097244\, y_4 + 197849421\, y_5 + 1272087803\, y_6 + 1950925264,$

$x_4 y_6^2 + 2000108329\, x_4 y_5 + 138882411\, x_5 y_5 + 1964621882\, y_4 x_6 + 1562649152\, y_5 x_6 + 274800980\, x_3 y_6 + 381168929\, x_4 y_6 + 1561080504\, x_5 y_6 + 646135501\, x_6 y_6 +$
$1252024999\, x_3 + 1828948462\, x_4 + 1907059409\, x_5 + 1062878925\, x_6,\ x_3 y_6^2 + 1940064434\, x_4 y_5 + 1699323466\, x_5 y_5 + 2767389\, y_4 x_6 + 309430653\, y_5 x_6 +$
$1746152111\, x_3 y_6 + 1486922955\, x_4 y_6 + 1042873400\, x_5 y_6 + 1877302158\, x_6 y_6 + 898857598\, x_3 + 2023749908\, x_4 + 1369459334\, x_5 + 1937240806\, x_6,$

$x_6^2 y_6 + 1859350309\, x_3 x_4 + 828165967\, x_3 x_5 + 1319416915\, y_4 y_5 + 1281531769\, y_5^2 + 416445396\, x_6^2 + 555896977\, y_4 y_6 + 838162654\, y_5 y_6 + 1094699319\, y_6^2 +$
$1025635396\, y_4 + 758820774\, y_5 + 1932663106\, y_6 + 902372666,\ y_5 x_6 y_6 + 1776737250\, x_4 y_5 + 1335234339\, x_5 y_5 + 197659465\, y_4 x_6 + 388691694\, y_5 x_6 +$
$1214819713\, x_3 y_6 + 1236939013\, x_4 y_6 + 1895585096\, x_5 y_6 + 1457663787\, x_6 y_6 + 1908824636\, x_3 + 1937866443\, x_4 + 906541898\, x_5 + 1779256072\, x_6,$

$y_4 x_6 y_6 + 392800087\, x_4 y_5 + 43314235\, x_5 y_5 + 1752015765\, y_4 x_6 + 697637736\, y_5 x_6 + 1174862040\, x_3 y_6 + 1726470482\, x_4 y_6 + 524280549\, x_5 y_6 + 1783594194\, x_6 y_6 +$
$777027038\, x_3 + 1196924612\, x_4 + 669351278\, x_5 + 136564514\, x_6,\ y_5^2 y_6 + 769157270\, x_3 x_4 + 30129177\, x_3 x_5 + 147541859\, y_4 y_5 + 696342885\, y_5^2 + 953052903\, x_6^2 + 63094058\, y_4 y_6 +$
$1607776536\, y_5 y_6 + 2003959420\, y_6^2 + 1657122998\, y_4 + 1041341194\, y_5 + 643382090\, y_6 + 298205040,\ x_5 y_5 y_6 + 1361368571\, x_4 y_5 + 443005480\, x_5 y_5 + 749246637\, y_4 x_6 +$
$556781711\, y_5 x_6 + 268588588\, x_3 y_6 + 179323388\, x_4 y_6 + 260672145\, x_5 y_6 + 542764427\, x_6 y_6 + 2031844241\, x_3 + 112806238\, x_4 + 2024966158\, x_5 + 1634398898\, x_6,$

$y_4 y_5 y_6 + 1495723262\, x_3 x_4 + 1150552515\, x_3 x_5 + 647627904\, y_4 y_5 + 834052394\, y_5^2 + 680400990\, x_6^2 + 703082161\, y_4 y_6 + 1261907640\, y_5 y_6 + 1146980666\, y_6^2 +$
$339024153\, y_4 + 1829077048\, y_5 + 1120614065\, y_6 + 420646718,\ x_4 y_5 y_6 + 391789202\, x_4 y_5 + 1778622432\, x_5 y_5 + 32574434\, y_4 x_6 + 638884222\, y_5 x_6 +$
$2008976092\, x_3 y_6 + 1158838637\, x_4 y_6 + 298082231\, x_5 y_6 + 579017100\, x_6 y_6 + 541015847\, x_3 + 1347513279\, x_4 + 1774560872\, x_5 + 1614705109\, x_6,$

$x_3 x_5 y_6 + 1581681716\, x_3 x_4 + 486946881\, x_3 x_5 + 421622009\, y_4 y_5 + 1075313850\, y_5^2 + 1564800523\, x_6^2 + 198951616\, y_4 y_6 + 1466002977\, y_5 y_6 + 932669036\, y_6^2 + 248319512\, y_4 +$
$862020011\, y_5 + 649537600\, y_6 + 815933435,\ x_3 x_4 y_6 + 1343545648\, x_3 x_4 + 1023324514\, x_3 x_5 + 40371239\, y_4 y_5 + 1905289341\, y_5^2 + 1639954889\, x_6^2 + 786545101\, y_4 y_6 +$
$1219192433\, y_5 y_6 + 321512152\, y_6^2 + 1631897898\, y_4 + 850776521\, y_5 + 530499711\, y_6 + 2036743747,\ x_6^3 + 1359379754\, x_4 y_5 + 4699239\, x_5 y_5 + 1446967796\, y_4 x_6 +$
$260472488\, y_5 x_6 + 701675423\, x_3 y_6 + 1889155319\, x_4 y_6 + 112548169\, x_5 y_6 + 1629096917\, x_6 y_6 + 658508665\, x_3 + 820850436\, x_4 + 665336977\, x_5 + 1707190979\, x_6,$

$y_5 x_6^2 + 1013046601\, x_3 x_4 + 969596453\, x_3 x_5 + 1553889292\, y_4 y_5 + 1185309841\, y_5^2 + 1987921573\, x_6^2 + 1033458441\, y_4 y_6 + 1320068753\, y_5 y_6 + 1102491211\, y_6^2 +$
$1104911459\, y_4 + 1375116864\, y_5 + 672833739\, y_6 + 626376074,\ y_4 x_6^2 + 2009134087\, x_3 x_4 + 1611713763\, x_3 x_5 + 168461479\, y_4 y_5 + 1706153267\, y_5^2 +$
$1769015690\, x_6^2 + 1182579576\, y_4 y_6 + 557864255\, y_5 y_6 + 503714053\, y_6^2 + 176291393\, y_4 + 1354065871\, y_5 + 954347026\, y_6 + 734410570,$

$y_5^2 x_6 + 619010252\, x_4 y_5 + 916121455\, x_5 y_5 + 1431371638\, y_4 x_6 + 969212309\, y_5 x_6 + 1949990023\, x_3 y_6 + 414782496\, x_4 y_6 + 1907745509\, x_5 y_6 + 970368126\, x_6 y_6 +$
$1740320236\, x_3 + 1975330810\, x_4 + 2143293978\, x_5 + 252311982\, x_6,\ y_4 y_5 x_6 + 558167487\, x_4 y_5 + 433016430\, x_5 y_5 + 2075138717\, y_4 x_6 + 1434835475\, y_5 x_6 +$
$531264210\, x_3 y_6 + 427467244\, x_4 y_6 + 1374860777\, x_5 y_6 + 149117380\, x_6 y_6 + 1826680361\, x_3 + 969629736\, x_4 + 766694650\, x_5 + 1666548268\, x_6,$

$y_5^3 + 649714439\, x_3 x_4 + 1076476457\, x_3 x_5 + 1435812662\, y_4 y_5 + 2053151093\, y_5^2 + 280374149\, x_6^2 + 1469939973\, y_4 y_6 + 1400337770\, y_5 y_6 + 1634063342\, y_6^2 +$
$354162717\, y_4 + 737861553\, y_5 + 816931778\, y_6 + 1428529698,\ x_5 y_5^2 + 1136639110\, x_4 y_5 + 121108532\, x_5 y_5 + 2127022098\, y_4 x_6 + 701800649\, y_5 x_6 +$
$1281723728\, x_3 y_6 + 2092528324\, x_4 y_6 + 1816317333\, x_5 y_6 + 1524717023\, x_6 y_6 + 737384683\, x_3 + 261085830\, x_4 + 712596842\, x_5 + 1219275979\, x_6,$

$y_4 y_5^2 + 1382099903\, x_3 x_4 + 1674451197\, x_3 x_5 + 1964164303\, y_4 y_5 + 610824582\, y_5^2 + 1726175807\, x_6^2 + 1045412838\, y_4 y_6 + 1328732288\, y_5 y_6 + 1416893499\, y_6^2 +$
$509989107\, y_4 + 356562705\, y_5 + 701591991\, y_6 + 90791056,\ x_4 y_5^2 + 1125381690\, x_4 y_5 + 343309511\, x_5 y_5 + 412315532\, y_4 x_6 + 392837310\, y_5 x_6 +$
$1859774430\, x_3 y_6 + 1289634195\, x_4 y_6 + 511405427\, x_5 y_6 + 2104680646\, x_6 y_6 + 1304660656\, x_3 + 1431387822\, x_4 + 2142663821\, x_5 + 395031648\, x_6\}$

Determine the Number of Solutions

Leading monomials:

y_3	x_5x_6	x_4x_6	x_3x_6	x_3y_5	x_5^2	y_4x_5	x_4x_5
y_4^2	x_4y_4	x_3y_4	x_4^2	x_3^2	y_6^3	$x_6y_6^2$	$y_5y_6^2$
$x_5y_6^2$	$y_4y_6^2$	$x_4y_6^2$	$x_3y_6^2$	$x_6^2y_6$	$y_5x_6y_6$	$y_4x_6y_6$	$y_5^2y_6$
$x_5y_5y_6$	$y_4y_5y_6$	$x_4y_5y_6$	$x_3x_5y_6$	$x_3x_4y_6$	x_6^3	$y_5x_6^2$	$y_4x_6^2$
$y_5^2x_6$	$y_4y_5x_6$	y_5^3	$x_5y_5^2$	$y_4y_5^2$	$x_4y_5^2$		

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$x_5y_6^2$	$y_4y_6^2$	$x_4y_6^2$	$x_3y_6^2$	$x_6^2y_6$	$y_5x_6y_6$	$y_4x_6y_6$	$y_5^2y_6$
$x_5y_5y_6$	$y_4y_5y_6$	$x_4y_5y_6$	$x_3x_5y_6$	$x_3x_4y_6$	x_6^3	$y_5x_6^2$	$y_4x_6^2$
$y_5^2x_6$	$y_4y_5x_6$	y_5^3	$x_5y_5^2$	$y_4y_5^2$	$x_4y_5^2$		

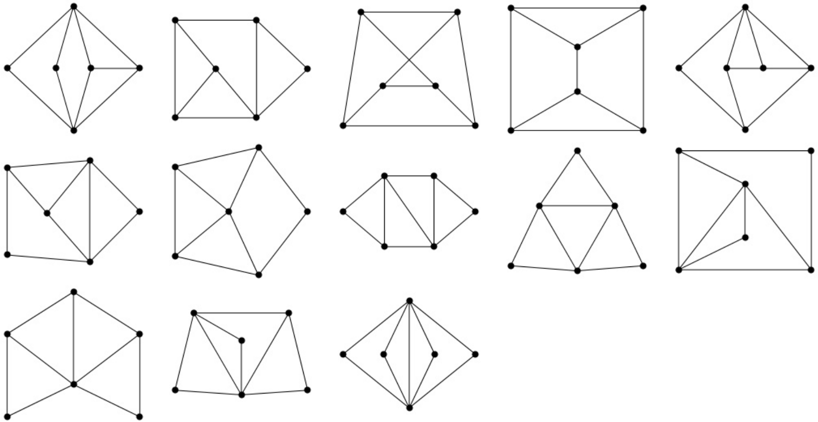
Monomials under the staircase:

1	y_6	x_6	y_5	x_5	y_4	x_4	x_3
y_6^2	x_6y_6	y_5y_6	x_5y_6	y_4y_6	x_4y_6	x_3y_6	x_6^2
y_5x_6	y_4x_6	y_5^2	x_5y_5	y_4y_5	x_4y_5	x_3x_5	x_3x_4

$\longrightarrow$ 24 complex solutions.

Laman Numbers

All but one m.r. graphs with 6 vertices have Laman number 16.



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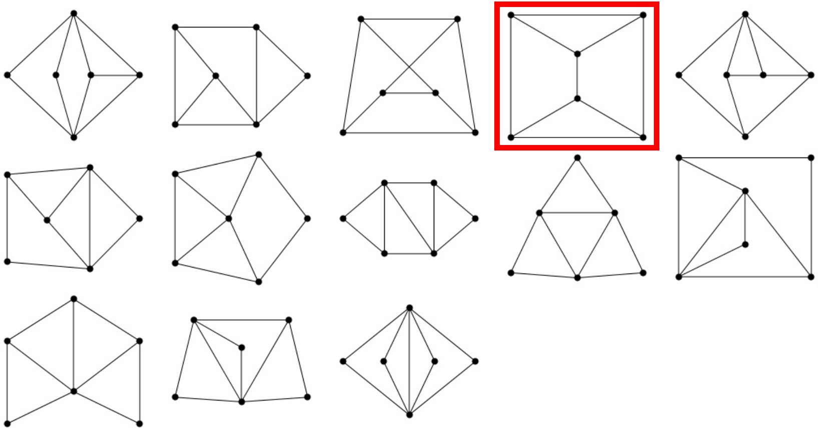

The only exception is the three-prism graph with $\mathrm{Lam}(\blacksquare) = 24$.

Laman Number as Degree

Recall: For each edge $\{u, v\} \in E$ we get an equation

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(u, v)^2.$$

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i.e., by the number how often a generic $\big(\lambda(u,v)\big)_{\{u,v\}\in E} \in \mathbb{C}^E$ is hit by the map f_G (modulo translations and rotations).

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Strategy: Apply methods from **algebraic geometry**.

- Work in projective space.
- f_G then should be a homogeneous map.

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Hence our map becomes

$$\begin{aligned} f_G\colon\ & \mathbb{C}^V \times \mathbb{C}^V \to \mathbb{C}^E, \\ & (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \big((x_u - x_v) \cdot (y_u - y_v)\big)_{\{u,v\} \in E} \end{aligned}$$

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Proposition: The Laman number $\mathrm{Lam}(G)$ of $G = (V, E)$ is given by the degree of the map

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Definition: A **bigraph** $B = (G, H)$ is a pair of graphs $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$, allowing several components, multiple edges and self-loops. The set $\mathcal{E}$ is called the set of **biedges**.

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Proposition: For $B = (G, G)$ we have $\mathrm{Lam}(B) = \mathrm{Lam}(G)$.

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- Apply **tropicalization**: Let d_V (resp. d_W) be the valuations of the x- (resp. y-) coordinates in the preimage of $\lambda_e s^{\mathrm{wt}(e)}$.
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- Each bidistance can be characterized by a single 0/1-vector.
- The set of preimages is **partitioned** w.r.t. the bidistances:

$$\mathrm{Lam}(B) = \sum_d \mathrm{Lam}(B_d).$$

jump

Puiseux Series

- $\mathbb{K} = \mathbb{C}\{\{s\}\}$: field of Puiseux series with coefficients in $\mathbb{C}$
- This field comes with a valuation $\nu \colon \mathbb{K} \setminus \{0\} \longrightarrow \mathbb{Q}$:

$$\nu\left(\sum_{i=k}^{+\infty} c_i \, s^{i/n}\right) = \frac{k}{n} \qquad \text{if } c_k \neq 0,$$

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For the map $f_{B,\mathbb{K}}\colon \mathbb{P}^{(\dots)}_{\mathbb{K}} \times \mathbb{P}^{(\dots)}_{\mathbb{K}} \longrightarrow \mathbb{P}^{|\mathcal{E}|-1}_{\mathbb{K}}$, obtained as the extension of scalars from f_B, we have $\deg(f_{B,\mathbb{K}}) = \deg(f_B)$.

Study the preimage of a "perturbed" point in $\mathbb{P}^{|\mathcal{E}|-1}_{\mathbb{K}}$:

$$f^{-1}_{B,\mathbb{K}}\Big(\big(\lambda_e s^{\mathrm{wt}(e)}\big)_{e\in\mathcal{E}}\Big) \quad \text{for some } \mathrm{wt} \in \mathbb{Q}^{\mathcal{E}} \text{ and } \lambda \in \mathbb{C}^{\mathcal{E}},$$

instead of studying the preimage $f_B^{-1}(p)$ for some $p \in \mathbb{P}^{|\mathcal{E}|-1}_{\mathbb{C}}$.

New Coordinates, New Equations

Introduce new coordinates

- x_{uv} for all $u, v \in V$ that are connected by an edge in G
- y_{tw} for all $t, w \in W$ that are connected by an edge in H

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Select a distinguished biedge $\bar{e} \in \mathcal{E}$. Then these coordinates satisfy the system of equations:

$$
\begin{aligned}
x_{\bar{u}\bar{v}} = y_{\bar{t}\bar{w}} &= 1 & \\
x_{uv}\, y_{tw} &= \lambda_e s^{\mathrm{wt}(e)} & \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\} \\
\sum\nolimits_{\mathscr{C}} x_{uv} &= 0 & \text{for all cycles } \mathscr{C} \text{ in } G \\
\sum\nolimits_{\mathscr{D}} y_{tw} &= 0 & \text{for all cycles } \mathscr{D} \text{ in } H
\end{aligned}
$$

In particular, $x_{uv} = -x_{vu}$.

Tropicalization

Goal: For a fixed point $p = \left(\lambda_e \, s^{\mathrm{wt}(e)}\right)_{e \in \mathcal{E}} \in \mathbb{P}_{\mathbb{K}}^{|\mathcal{E}|-1}$ we want to determine its preimages $f_{B,\mathbb{K}}^{-1}(p)$.

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Idea:

- Apply **tropicalization**: look only at the valuations!
- An algebraic relation between Puiseux series implies a piecewise linear relation between their orders.
- For $q \in f_{B,\mathbb{K}}^{-1}(p)$ let $d_V(u,v) = \nu(q_{x_{uv}})$, $d_W(t,w) = \nu(q_{y_{tw}})$.
- This way we obtain a discrete object, a pair of functions (d_V, d_W), that we call **bidistance**.

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- This way we obtain a discrete object, a pair of functions (d_V, d_W), that we call **bidistance**.

Gain: We can then partition the set $f_{B,\mathbb{K}}^{-1}(p)$ according to the bidistances that are determined by its elements.

Bidistances

The functions d_V and d_W satisfy

- $d_V(u,v) = d_V(v,u)$ for all (u,v), and similarly for d_W
- $d_V(u,v) + d_W(t,w) = \mathrm{wt}(e)$ for all $e \in \mathcal{E} \setminus \{\bar{e}\}$
- $d_V(\bar{u},\bar{v}) = d_W(\bar{t},\bar{w}) = 0$
- for every cycle $\mathscr{C}$ in G, the minimum of the values of d_V on the pairs of vertices (u,v) appearing in $\mathscr{C}$ is attained at least twice, and similarly for d_W.

Definition: Every pair of functions (d_V, d_W) satisfying the above conditions is called a **bidistance** compatible with $\mathrm{wt} \in \mathbb{Q}^{|\mathcal{E}|-1}$.

Recursion for the Laman number

Idea: We partition the set $f_{B,\mathbb{K}}^{-1}(p)$ according to the bidistances.

Lemma: The number of preimages sharing the same bidistance d can be obtained as the Laman number of a “simpler” Graph B_d.

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Unfortunately, it is not very useful for practical purposes:

1. Enumeration of bidistances d: **difficult**
2. Computation of $\mathrm{Lam}(B_d)$: **difficult**

Two specializations in order to get more explicit formulas...

First Strategy

By choosing a general weight vector $\mathrm{wt} \in \mathbb{Q}^{|\mathcal{E}|-1}$, one can show that $\mathrm{Lam}(B_d) = 1$ for every bidistance d compatible with wt.

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The computation of $\mathrm{Lam}(B)$ is therefore reduced to a piecewise linear problem:

1. Enumeration of bidistances d: **difficult**
2. Computation of $\mathrm{Lam}(B_d)$: **trivial**

Second Strategy

Idea: We choose the special weight vector $(1, \ldots, 1) \in \mathbb{Q}^{|\mathcal{E}|-1}$.

We can show that in this case the values of d_V and d_W are

- integers
- moreover: only the values 0 and 1 can occur.

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We can show that in this case the values of d_V and d_W are

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- moreover: only the values 0 and 1 can occur.

Hence, each bidistance can be characterized by a single vector in $\{0, 1\}^{|\mathcal{E}|-1}$ (since $d_V + d_W = 1$ for all $e \in \mathcal{E} \setminus \{\bar{e}\}$).

1. Enumeration of bidistances d: **easy**
2. Computation of $\mathrm{Lam}(B_d)$: **feasible**

Operations on Graphs

For constructing the graph B_d, we need to introduce two operations on graphs:

- complement
- quotient

Graph Complement

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph complement** $G \setminus E'$ is defined as

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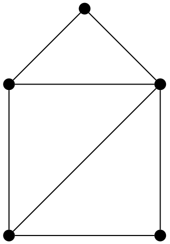

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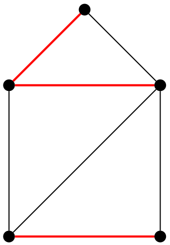

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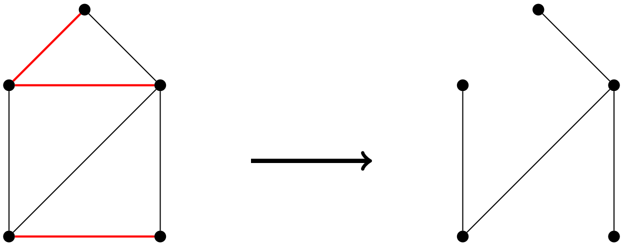

Graph Quotient

Let $G = (V, E)$ be a graph and let $E' \subseteq E$.

Definition: The **graph quotient** $G \,/\, E'$ is constructed as follows:

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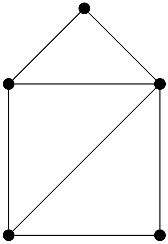

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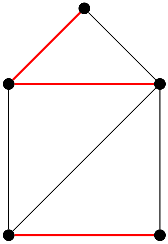

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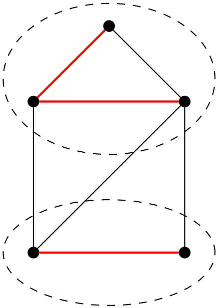

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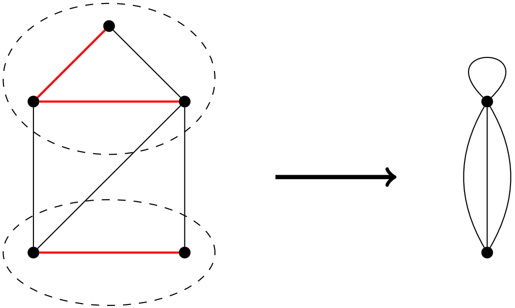

Operations on Bigraphs

We define the following two operations on a bigraph $B = (G, H)$:
For a subset $\mathcal{M} \subseteq \mathcal{E}$ of the biedges $\mathcal{E}$ let

- ${}^{\mathcal{M}}B := (G \,/\, \mathcal{M},\ H \setminus \mathcal{M})$
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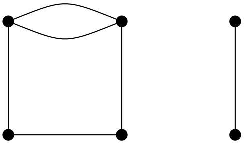
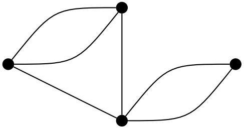


$H =$



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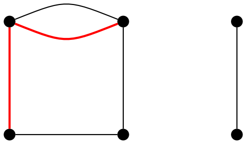

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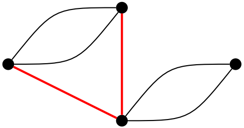

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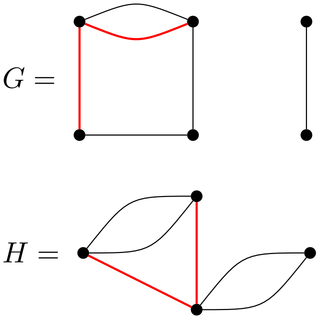

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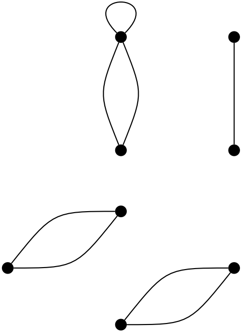

The bigraph ${}^{\mathcal{M}}B$

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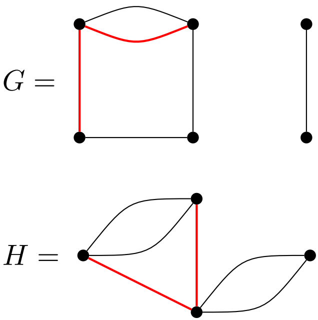

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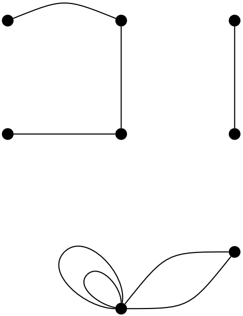

The bigraph $B^{\mathcal{M}}$

The Combinatorial Algorithm

Theorem. Let $B = (G, H)$ be a bigraph with $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. Choose $\bar{e} \in \mathcal{E}$. Then

$$\mathrm{Lam}(B) = \mathrm{Lam}\left({}^{\{\bar{e}\}}B\right) + \mathrm{Lam}\left(B^{\{\bar{e}\}}\right) + \sum_{\substack{\mathcal{M} \cup \mathcal{N} = \mathcal{E} \\ \mathcal{M} \cap \mathcal{N} = \{\bar{e}\}}} \mathrm{Lam}\left({}^{\mathcal{M}}B\right) \cdot \mathrm{Lam}\left(B^{\mathcal{N}}\right).$$

Initial conditions:

- $\mathrm{Lam}(G) = \mathrm{Lam}(G, G)$
- $\mathrm{Lam}(B) = 0$ if G or H contains a loop.
- $\mathrm{Lam}(B) = 0$ if $|V| - |\operatorname{Comp}(G)| + |W| - |\operatorname{Comp}(H)| \neq |\mathcal{E}| + 1$.
- $\mathrm{Lam}(B) = 1$ if $|\mathcal{E}| = 1$ and if there are no loops.

Minimally Rigid Graphs with Most Realizations

Question: Among all minimally rigid graphs with n vertices, which one has the largest number of realizations?

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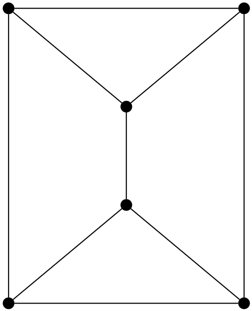
n	6
#	24



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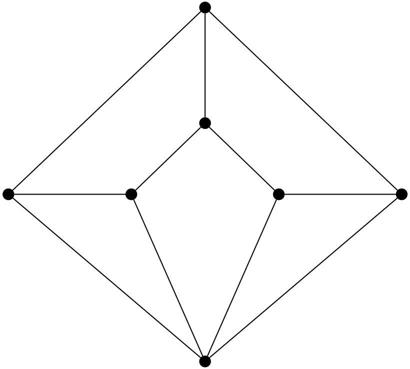
Question: Among all minimally rigid graphs with n vertices, which one has the largest number of realizations?

n	6	7
#	24	56



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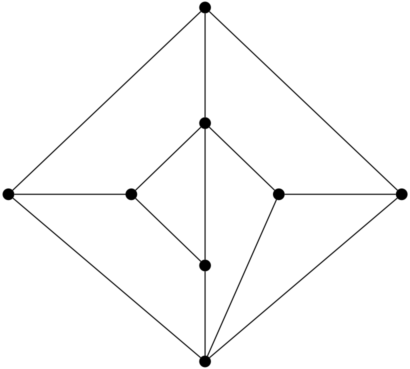
Question: Among all minimally rigid graphs with n vertices, which one has the largest number of realizations?

n	6	7	8
#	24	56	136



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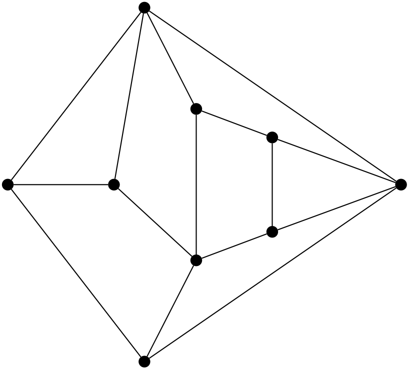
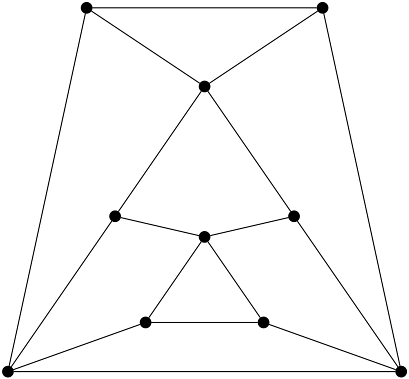
n	6	7	8	9
#	24	56	136	344



Minimally Rigid Graphs with Most Realizations

Question: Among all minimally rigid graphs with n vertices, which one has the largest number of realizations?

n	6	7	8	9	10
$\#$	24	56	136	344	880



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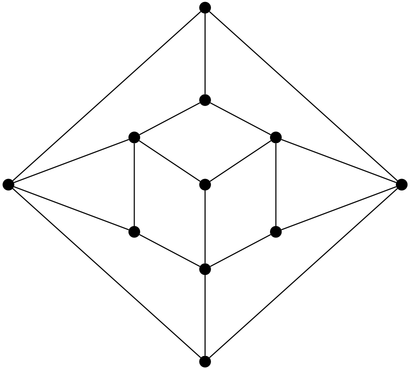
n	6	7	8	9	10	11
#	24	56	136	344	880	2288



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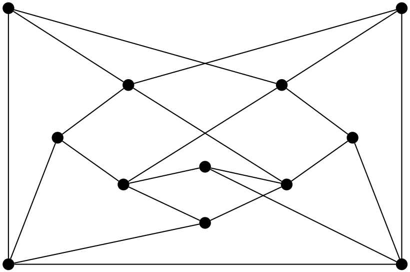
Question: Among all minimally rigid graphs with n vertices, which one has the largest number of realizations?

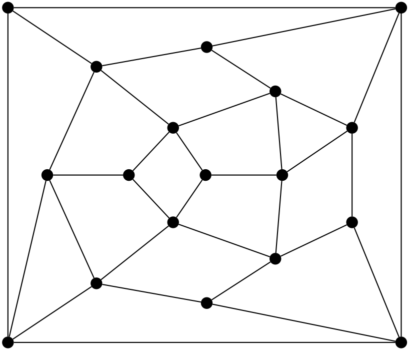
n	6	7	8	9	10	11	12
#	24	56	136	344	880	2288	6180



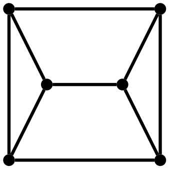
Minimally Rigid Graphs with Most Realizations

Question: Among all minimally rigid graphs with n vertices, which one has the largest number of realizations?

n	6	7	8	9	10	11	12		18
#	24	56	136	344	880	2288	6180	...	$\geqslant 1953816$

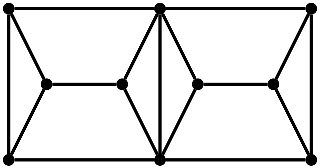


Caterpillar Construction



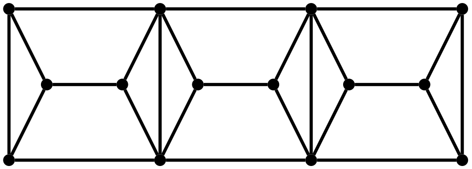
- Choose a m.r. graph $G = (V, E)$ (e.g.: three-prism graph).

Caterpillar Construction



- Choose a m.r. graph $G = (V, E)$ (e.g.: three-prism graph).
- Place k copies of G and connect them with shared edges.

Caterpillar Construction



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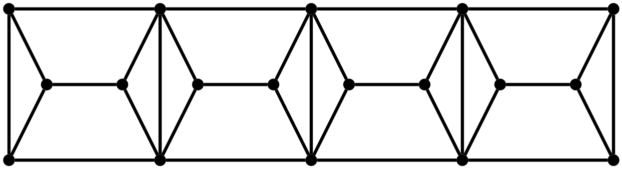

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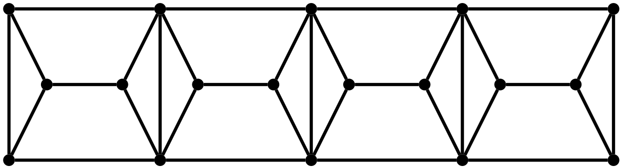

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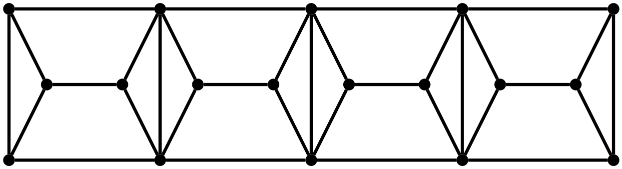
- Place k copies of G and connect them with shared edges.
- One gets $2 + k \cdot (|V| - 2)$ vertices and $1 + k \cdot (|E| - 1)$ edges.

Caterpillar Construction



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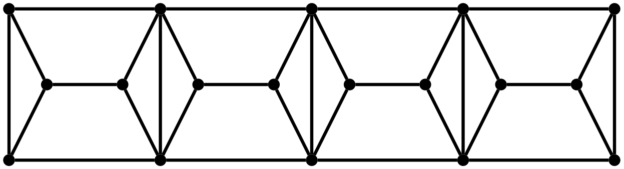

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Hence, for any minimally rigid graph G and $n \geqslant 2$,
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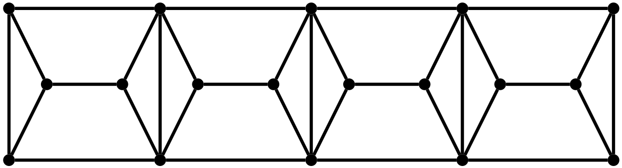
$$\mathrm{Lam}(G)^{\lfloor (n-2)/(|V|-2) \rfloor}.$$

Caterpillar Construction



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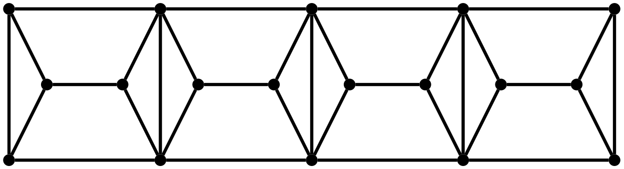

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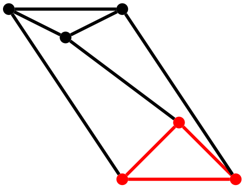
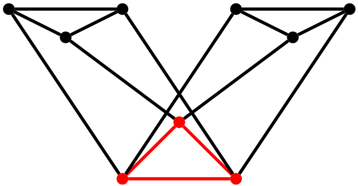
Growth rate using the three-prism graph: $24^{n/4} \approx 2.21336^n$.

Fan Construction



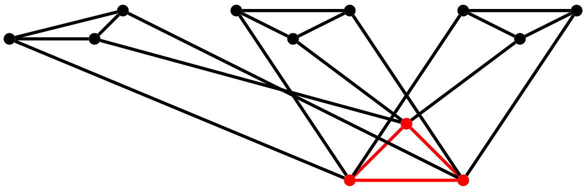
- Choose a m.r. graph $G = (V, E)$ containing a triangle H.

Fan Construction



- Choose a m.r. graph $G = (V, E)$ containing a triangle H.
- Place k copies of G sharing this triangle $H = (W, F)$.

Fan Construction



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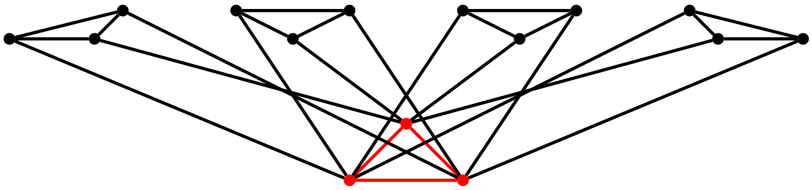

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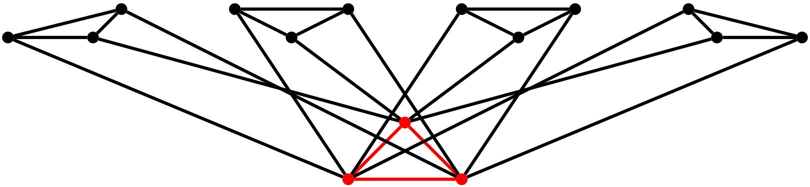
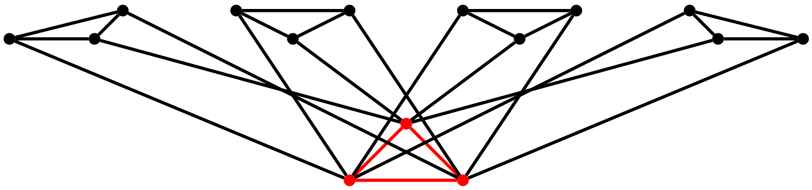
Fan Construction



- Choose a m.r. graph $G = (V, E)$ containing a triangle H.
- Place k copies of G sharing this triangle $H = (W, F)$.
- One gets $3 + k \cdot (|V| - 3)$ vertices and $3 + k \cdot (|E| - 3)$ edges.

Fan Construction



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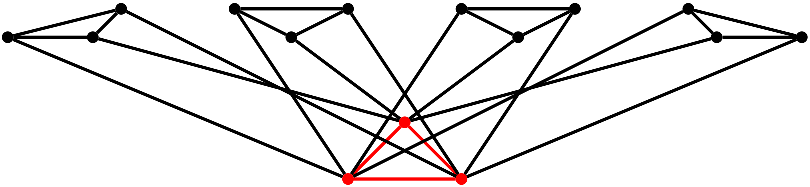
- Place k copies of G sharing this triangle $H = (W, F)$.
- One gets $3 + k \cdot (|V| - 3)$ vertices and $3 + k \cdot (|E| - 3)$ edges.
- Resulting graph has Laman number $2 \cdot \big(\mathrm{Lam}(G)/2\big)^k$.

Fan Construction



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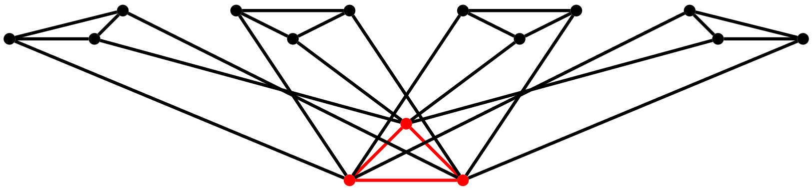
Hence, for any minimally rigid graph G and $n \geqslant 3$, there exists an n-vertex graph with realizations at least

$$2 \cdot \left(\frac{\mathrm{Lam}(G)}{2}\right)^{\lfloor (n-3)/(|V|-3) \rfloor}$$

Fan Construction



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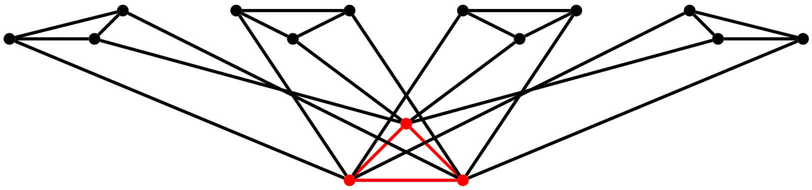
Fan Construction



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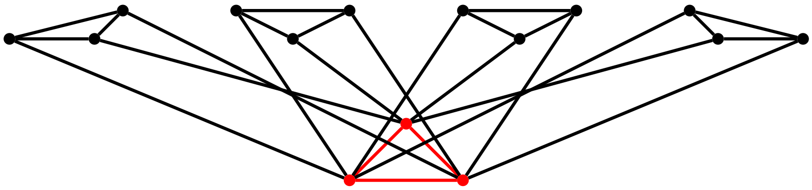
Growth rate using the three-prism graph: $12^{n/3} \approx 2.28943^n$.

Fan Construction

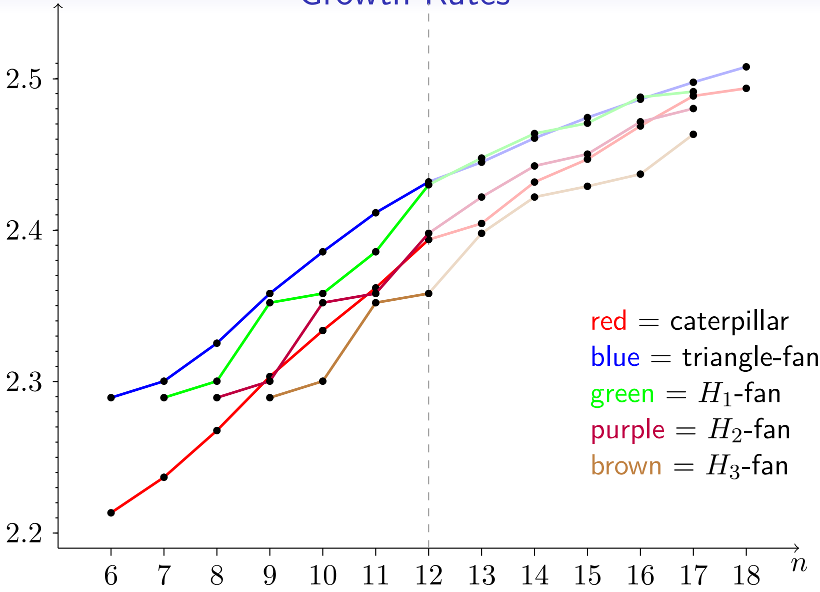


- Choose a m.r. graph $G = (V, E)$ containing a subgraph H.
- Place k copies of G sharing this m.r. subgraph $H = (W, F)$.
- $|W| + k \cdot (|V| - |W|)$ vertices and $|F| + k \cdot (|E| - |F|)$ edges.
- Resulting graph: $\mathrm{Lam} \geqslant \mathrm{Lam}(H) \cdot \left(\mathrm{Lam}(G)/\,\mathrm{Lam}(H)\right)^k$.

Hence, for any minimally rigid graph G and $n \geqslant |W|$,
there exists an n-vertex graph with realizations at least

$$2^{(n-|W|) \bmod (|V|-|W|)} \cdot \mathrm{Lam}(H) \cdot \left(\frac{\mathrm{Lam}(G)}{\mathrm{Lam}(H)}\right)^{\lfloor (n-|W|)/(|V|-|W|) \rfloor}$$

Growth Rates



Real realizations

Question: Given a m.r. graph G, can we find a **real** labeling $\lambda\colon E \to \mathbb{R}_{>0}$ such that there exist $\mathrm{Lam}(G)$ **real** embeddings?

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Answer: Sometimes, but not always.

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Example: