Geometric Axioms for Minkowski Spacetime and Without-Loss-of-Generality Theorems

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Outline

Approaches to spacetime

Formalisation in Isabelle/HOL

- Axioms
- Theorems

Symmetries and reasoning
without loss of generality

Approaches to spacetime

Spacetime as a metric space

In Special Relativity, spacetime is defined as $\mathbb{R}^4$ with a Minkowski metric.

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Advantage: Many things are easy to compute, since we can always assign coordinates and do linear algebra.

Disadvantage: Axioms rely on a hefty baggage of mathematical analytical foundations, hard to reconcile with experience.

Spacetime as an ordered geometry

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- more similar to Hilbert's *Grundlagen* in Euclidean geometry
- axioms closer to physical intuition (hopefully)

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Our primitives are the following.

Set of events $\mathcal{E}$

Set of paths $\mathcal{P}$

Betweenness $[\_\ \_\ \_]$

Formalisation in Isabelle/HOL

Prose to Isabelle/HOL

Proof (By induction). The previous theorem applies to the case where $n = 4$. We will make the inductive hypothesis that the result applies to a set of n distinct events $\{a_1, a_2, \dots, a_n\}$ and demonstrate that this implies the result for the case of $n+1$ distinct events. We denote the $(n+1)$–th event as b. Then Axiom O5 implies that either:

(i) $[ba_1a_n]$ or (ii) $[a_1ba_n]$ or (iii) $[a_1a_nb]$

Case (i): By the inductive hypothesis and Theorem 2 we have $[a_1a_2a_n]$ so the previous theorem (Th.9) implies that $[ba_1a_2a_n]$ which implies that $[ba_1a_2]$. Thus b is an element of a chain $[a_1^*a_2^* \dots a_{n+1}^*]$ where $a_1^* := b$ and (for $j \in \{2, \dots, n+1\}$) $a_j^* := a_{j-1}$.

Case (ii): Let k be the smallest integer such that $[a_1ba_k]$. Then the previous theorem (Th.9) implies either that $[a_1a_{k-1}ba_k]$, or that $k = 2$ so that $[a_{k-1}ba_k]$. If $k-2 \geq 1$ we have $[a_{k-2}a_{k-1}a_k]$ which with $[a_{k-1}ba_k]$ implies $[a_{k-2}a_{k-1}ba_k]$ by the previous theorem, while if $k+1 \leq n$ we have $[a_{k-1}a_ka_{k+1}]$ which with $[a_{k-1}ba_k]$ implies $[a_{k-1}ba_ka_{k+1}]$; that is we have now shown that $[a_{k-2}a_{k-1}b]$ (if $k-2 \geq 1$) and $[a_{k-1}ba_k]$ and $[ba_ka_{k+1}]$ (if $k+1 \leq n$) so that b is an element of a chain $[a_1^*a_2^* \dots a_{n+1}^*]$ where

$$a_j^* = \begin{cases} a_j, & j \leq k-1 \\ b, & j = k \\ a_{j-1}, & j > k\ . \end{cases}$$

Case (iii): The proof for this case is similar to that for Case (i).

```
text ‹This is case (i) of the induction in Theorem 10.›
lemma (*for 10*) chain_append_at_left_edge: [95 lines]

lemma (*for 10*) chain_append_at_right_edge: [61 lines]

lemma S_is_dense: [28 lines]

lemma (*for 10*) smallest_k_ex: [152 lines]

lemma get_closest_chain_events: [102 lines]

text ‹This is case (ii) of the induction in Theorem 10.›
lemma (*for 10*) chain_append_inside: [248 lines]
```

```
subsection "WLOG for two general symmetric relations of two
context MinkowskiBetweenness begin [241 lines]

subsection "WLOG for two intervals"
context MinkowskiBetweenness begin [78 lines]
```

Proof (i) Theorem 4 implies that both sets $Q(a, \emptyset)$ and $Q(b, \emptyset)$ are bounded in both directions by events which do not belong to the unreachable sets themselves, so the union $Q(a, \emptyset) \cup Q(b, \emptyset)$ is bounded by distinct events y, z which do not belong to the union of the unreachable sets.

```
lemma (*for 14i*) union_of_bounded_sets_is_bounded: [173 lines]
```

Order

O1 $[a\ b\ c] \implies \exists Q \in \mathcal{P} : a, b, c \in Q$

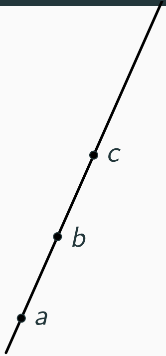
O2 $[a\ b\ c] \implies [c\ b\ a]$

O3 $[a\ b\ c] \implies a, b, c$ are distinct

O4 $[a\ b\ c] \wedge [b\ c\ d] \implies [a\ b\ d]$

O5 $a, b, c \in Q \implies a, b, c$ are ordered

O6 analogue of Pasch's axiom



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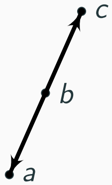

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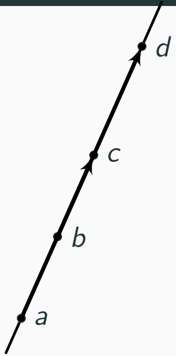

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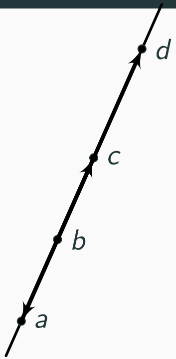

Definition (Chain)
A chain is a set of events $\{Q_i\}_{i \in I}$ with $I = \{0, 1, 2, ...\}$ such that

$$\forall i \in I. \quad i \geq 2 \implies [Q_{i-2}\ Q_{i-1}\ Q_i]\,.$$

Incidence (and Unreachable Sets)

I1 $\mathcal{E}$ is not empty.

I2 Distinct events are connected by intersecting paths.

I3 At most one path connects any two events.

I4 Axiom of Dimension

Incidence (and Unreachable Sets)

I5 Non-Galilean Axiom: 2 events in unreachable set

I6 Connectedness of the Unreachable Set

I7 Boundedness of the Unreachable Set

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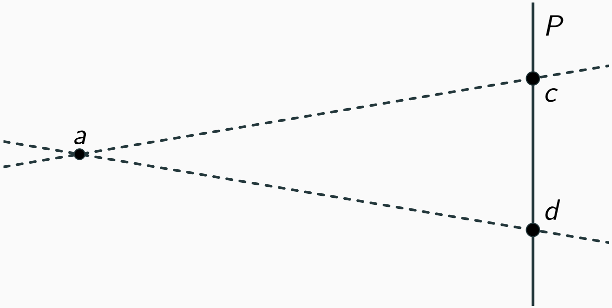
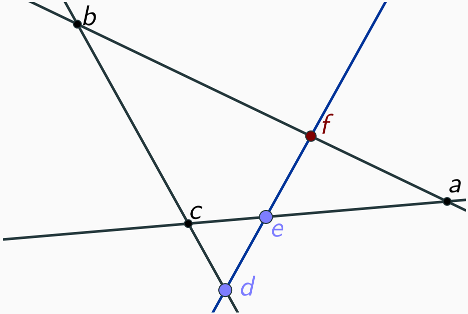


Collinearity and Infinity

Two Collinearity Theorems
to extend O6.

First CT: $[a\ f\ b]$

Second CT: $[d\ e\ f]$



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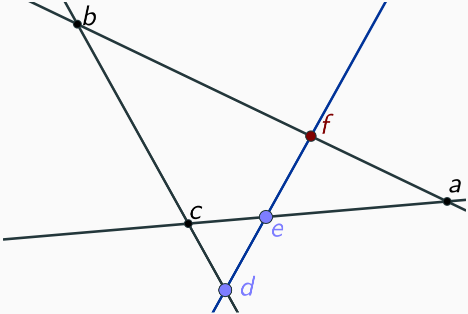


```
theorem (* 6 i i *) infinite_paths:
  assumes "P∈𝒫"
  shows "infinite P"
```

Chains, transitivity and linear order

A chain $\{Q_i\}_{i \in I}$ with $I = \{0, 1, 2, ...\}$ gives an index function

$$f : I \to \mathcal{E},\ i \mapsto Q_i \qquad \text{with } I \subseteq \mathbb{N}\ .$$

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```
theorem order_finite_chain2:
  assumes "long_ch_by_ord2 f X"
      and "finite X"
      and "0 ≤ i ∧ i < j ∧ j < l ∧ l < card X"
  shows "[[(f i) (f j) (f l)]]"
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```

```
theorem path_finsubset_chain:
  assumes "Q ∈ 𝒫"
      and "X ⊆ Q"
      and "card X ≥ 2"
  shows "ch X"
```

Symmetries and reasoning without loss of generality

Reversing chains 1

```
lemma chain_sym:
  assumes "[f[a..b..c]X]"
  shows "[λn. f (card X - 1 - n)[c..b..a]X]"
```

We use this lemma in proving linear order on paths (`path_finsubset_chain`):

1. inductively append an event e onto a chain $[f[a..b..c]X]$
2. consider cases $[e\ a\ b]$, $[a\ b\ e]$ (and $[a\ e\ b]$)

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2. consider cases $[e\ a\ b]$, $[a\ b\ e]$ (and $[a\ e\ b]$)
3. $f_2 : i \mapsto f(|X| - 1 - n)$
4. $g_2 : i \mapsto \begin{cases} b & \text{if } i = 0 \\ f_2(i-1) & \text{otherwise} \end{cases}$
5. $g : i \mapsto g_2(|X| - 1 - n)$

Reversing chains 2

```
lemma chain_unique_upto_rev:
  assumes "[f[a..c]X]" "[g[x..z]X]"
      and "card X ≥ 3" "i < card X"
  shows "f i = g i ∨ f i = g (card X - i - 1)"
```

- not present in the prose monograph
- makes it obvious there is more to chains than just their events
- used in an early proof of a theorem, “filling in” the original

Without Loss Of Generality (WLOG)

- Frequently used in pen-and-paper proofs,
  sometimes informally, often encompasses different symmetries
- Hard to mechanise:
  1. copy-paste-replace
  2. use variables or intermediate lemmas
  3. explicitly identify symmetries
- Our theory has several lemmas:
  1. for different levels of generality
  2. for different cases of distinctness and degeneracy

```
lemma linorder_less_wlog:
  assumes "⋀a b. P b a ⟹ P a b"
      and "⋀a. P a a"
      and "⋀a b. a < b ⟹ P a b"
  shows "P a b"
```

WLOG for interval endpoints 1

```
lemma wlog_interval_endpoints_distinct:
  assumes "⋀I J. ⟦is_int I; is_int J; P I J⟧ ⟹ P J I"
          "⋀I J a b c d. ⟦I = interval a b; J = interval c d⟧
          ⟹ (betw4 a b c d ⟶ P I J) ∧
             (betw4 a c b d ⟶ P I J) ∧
             (betw4 a c d b ⟶ P I J)"
  shows "⋀I J Q a b c d.
           ⟦I = interval a b; J = interval c d;
            I⊆Q; J⊆Q; Q∈𝒫;
            a≠b ∧ a≠c ∧ a≠d ∧ b≠c ∧ b≠d ∧ c≠d⟧
         ⟹ P I J"
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P is symmetric

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Essentially distinct orderings

WLOG for interval endpoints 2

Proofs mirror the structure of the preceding lemma.

1. State the desired result
2. Split up the proof into essentially distinct cases with fixed events

```
let ?prop = "λ I J. is_int (I∩J) ∨ (I∩J) = {}"
{ fix I J a b c d
  assume "I = interval a b" "J = interval c d"
  { assume "betw4 a b c d"
    have "I∩J = {}" ...
  } { assume "betw4 a c b d"
    have "I∩J = interval c b" ...
  } { assume "betw4 a c d b"
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  } }
then show "is_int (I1∩I2)"
  using wlog_interval_endpoints_distinct symmetry assms
  by simp
```

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References

[1] Richard Schmoetten, Jake Palmer, and Jacques Fleuriot. Schutz' independent axioms for Minkowski spacetime. *Archive of Formal Proofs*, July 2021. `https://isa-afp.org/entries/Schutz_Spacetime.html`, Formal proof development.

[2] Richard Schmoetten, Jake E. Palmer, and Jacques D. Fleuriot. Towards formalising Schutz' axioms for Minkowski spacetime in Isabelle/HOL. `http://arxiv.org/abs/2108.10868v2`.

[3] John W. Schutz. *Independent Axioms for Minkowski Space-Time*. CRC Press, October 1997.

Summary and Future Work

- We have formalised most of Chapter 3, several other lemmas.
- We can explicitly use symmetries to replace copy-paste proofs.
- WLOG lemmas can automate (to a degree)
  the switch from a symmetry to a sufficient list of cases.

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- WLOG lemmas can automate (to a degree)
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Directions to explore in the future:

- ▷ continue the mechanisation (Continuity, Chapter 4)
- ▷ extend generality of WLOG lemmas,
  identify content-independent aspects

Isotropy

If Q, R, S are distinct paths which meet at some event x and if $Q_a \in Q$ is an event distinct from x such that

$$Q(Q_a, R, x, \emptyset) = Q(Q_a, S, x, \emptyset)$$

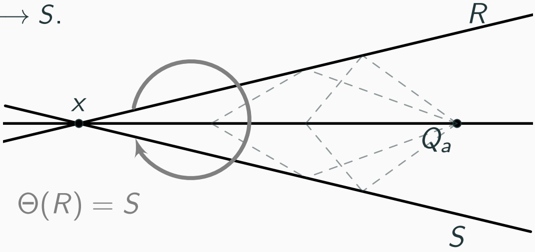
then

(i) there is a mapping $\theta : \mathcal{E} \longrightarrow \mathcal{E}$

(ii) which induces a bijection $\Theta : \mathcal{P} \longrightarrow \mathcal{P}$, such that

(iii) the events of Q are invariant, and

(iv) $\Theta : R \longrightarrow S$.



Continuity

Set of bounds $\mathcal{B} = \{Q_b : i < j \implies [Q_i\ Q_j\ Q_b]; Q_i, Q_j, Q_b \in Q\}$

Closest bound $Q_b \in \mathcal{B}$ such that for all $Q_{b'} \in \mathcal{B} \setminus \{Q_b\}$,

$$[Q_0\ Q_b\ Q_{b'}]$$

Continuity Any bounded infinite chain has a closest bound.

```
definition is_bound_f :: ... "is_bound_f Q_b Q f ≡
    ∀i j ::nat. [f[(f 0)..]Q] ∧
                (i<j ⟶ [[(f i) (f j) Q_b]])"
definition bounded :: ... "bounded Q ≡
    ∃ Q_b f. is_bound_f Q_b Q f"
definition closest_bound :: ... "closest_bound Q_b Q ≡
    ∃f. is_bound_f Q_b Q f ∧
        (∀ Q_b'. (is_bound Q_b' Q ∧ Q_b' ≠ Q_b)
                 ⟶ [[(f 0) Q_b Q_b']])"
```