Mechanization of incidence projective geometry in higher dimensions, a combinatorial approach

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Introduction

Short story

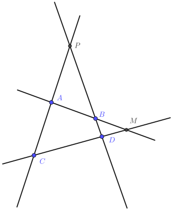
Few years ago, D. Michelucci and I wanted to have a fast automatic prover in order to avoid degenerate cases in a geometric constraints solving process.

- we focused on projective incidence geometry.
- we wanted to avoid coordinates and we studied some combinatorial methods, in particular matroid theory.
- in Strasbourg, we succeeded in proving Desargues's theorem with ranks and to have a certified proof in Coq.
- D. Braun, developed an automatic solver based on these ideas and succeeded in formally proving Dandelin-Gallucci's theorem.
- all our investigations concerned 2D and 3D, but it was possible to extend them toward higher dimensions.

Set of axioms

Axioms independent from dimension

1. $\forall\ A\ B : \text{Point}\ \exists\ d : \text{Line}, A \in d \wedge B \in d$
2. $\forall\ A\ B : \text{Point}\ \forall\ d\ d' : \text{Line}, A \in d \wedge B \in d \wedge A \in d' \wedge B \in d' \Rightarrow A = B \vee d = d'$
3. $\forall\ d : \text{Line}\ \exists\ A\ B\ C : \text{Point}, A \neq B \wedge A \neq B \wedge B \neq C \wedge A \in d \wedge B \in d \wedge C \in d$
4. $\forall A\ B\ C\ D\ M : \text{Point}\ \forall\ d_1\ d_2\ d_3\ d_4 :$ Line,
   $A \in d_1 \wedge B \in d_1 \wedge M \in d_1 \wedge$
   $C \in d_2 \wedge D \in d_2 \wedge M \in d_2 \wedge$
   $A \in d_3 \wedge C \in d_3 \wedge B \in d_4 \wedge D \in d_4$
   $\Rightarrow$
   $\exists\ P : \text{Point}, P \in d_3 \wedge P \in d_4$



Set of axioms (2)

Axioms for the plane

1. $\forall\ d\ \ d' : \text{Line}\ \exists\ A : \text{Point}, A \in d \wedge A \in d'$
2. $\exists\ d\ \ d' : \text{Line}, d \neq d'$

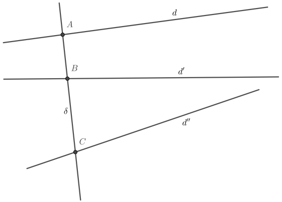
Set of axioms (2)

Axioms for the plane

1. $\forall\ d\ d'$: Line $\exists\ A$: Point, $A \in d \wedge A \in d'$
2. $\exists\ d\ d'$: Line, $d \neq d'$

(Usual) Axioms for the 3D-space

1. $\exists\ d\ d'$: Line, $\neg(\exists A$: Point, $A \in d\ \wedge\ A \in d')$
2. $\forall\ d\ d'\ d''$: Line
   $\exists\ A\ B\ C$: Point $\exists\ \delta$: Line,
   $A \in d \wedge A \in \delta \wedge$
   $B \in d' \wedge B \in \delta \wedge$
   $C \in d'' \wedge C \in \delta$



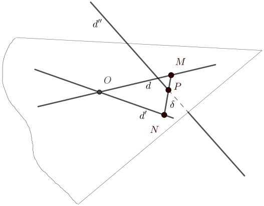
Set of axioms (3)

Axioms for the plane

1. $\forall\ d\ d' : \text{Line}\ \exists\ A : \text{Point}, A \in d \wedge A \in d'$
2. $\exists\ d\ d' : \text{Line}, d \neq d'$

(Alternate) Axioms for the 3D-space

1. $\exists\ d\ d' : \text{Line}, \neg(\exists A : \text{Point}, A \in d\ \wedge\ A \in d')$
2. $\forall\ d\ d'\ d'' : \text{Line}, \forall\ O : \text{Point}$
   $d \neq d'\ \wedge\ O \in d\ \wedge\ O \in d' \Rightarrow$
   $\exists PMN : \text{Point}, \exists \delta$
   $P \in d'' \wedge$
   $O \notin \delta \wedge P \in \delta$
   $M \in \delta \wedge M \in d \wedge$
   $N \in \delta \wedge N \in d'$

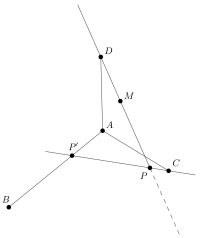


In n dimensions

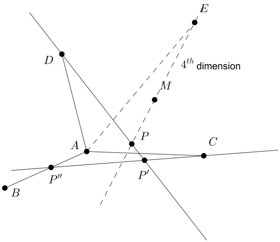
Idea

In dimension n, a hyperplane is a subspace (a flat) with dimension $n-1$.
Then, the upper-dimension axiom states that for any hyperplane H and any line δ, there is a point P belonging to H and δ.
$\Rightarrow$ inductive definition of n-dimensional flat and incidence point-flat.

In 3D



In 4D



Matroid theory (Whitney, 1935)

- Goal : axiomatically capture the notion of linear dependency (without coordinates) ...
- Lot of equivalent definitions:
  - independent or dependent sets
  - bases
  - closure
  - rank functions
  - ...
- the notion of rank function fits well to our context (and make defining dimensions easier)

Axioms for defining a rank function

Consider a set E and its powerset to which X and Y belong:

(Bounds)

$(A_1)\quad \forall\, X,\ 0 \leq \mathrm{rk}(X) \leq |X|$

(Monotonicity)

$(A_2)\quad \forall\, X\ Y,\ X \subseteq Y \Rightarrow \mathrm{rk}(X) \leq \mathrm{rk}(Y)$

(Submodularity)

$(A_3)\quad \forall\, X\ Y,\ \mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leq \mathrm{rk}(X) + \mathrm{rk}(Y)$

Geometric axioms

(A_4) $\forall P,\ \mathrm{rk}(\{P\}) = 1$

(A_5) $\forall P\ Q, P \neq Q \Rightarrow \mathrm{rk}(\{P, Q\}) = 2$

(A_6) $\forall\ A\ B\ C\ D,\ \mathrm{rk}(\{A, B, C, D\}) \leq 3\ \Rightarrow$
$\exists\ J :,\ \mathrm{rk}(\{A, B, J\}) = \mathrm{rk}(\{C, D, J\}) = 2$

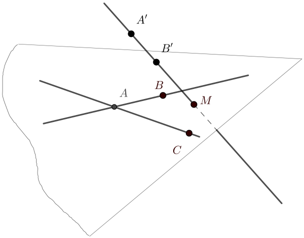
(A_7) $\forall\ A\ B,\ \exists\ C,$
$\mathrm{rk}(\{A, B, C\}) = \mathrm{rk}(\{B, C\}) = \mathrm{rk}(\{A, C\}) = 2$

Axioms for fixing a dimension (in 3D)

(A_8) $\exists A\ B\ C\ D,\ \text{rk}(\{A, B, C, D\}) \geq 4$

(A_9) $\forall A\ B\ C\ D,\ \text{rk}(\{A, B, C, D\}) \leq 4$

(A_{10}) $\forall\ A\ B\ C\ A'\ B', \exists\ M,$
$\text{rk}(\{A, B, C\}) = 3\ \wedge$
$\text{rk}(\{A', B'\}) = 2 \Rightarrow$
$\text{rk}(\{A, B, C, M\}) = 3\ \wedge$
$\text{rk}(\{A', B', M\}) = 2$



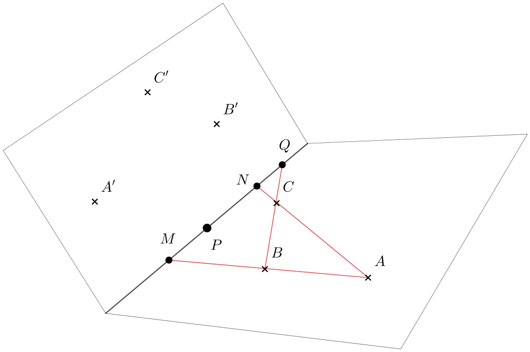
Result

In dimensions 2 and 3, the geometric axioms "are equivalent" to the corresponding ones expressed in matroid terms.

Utilization through a simple example

$$
\begin{array}{l}
\forall\ A\ B\ C, \forall\ A'\ B'\ C', \exists\ M\ N, \forall\ P\\
\mathrm{rk}(\{A,B,C\}) = 3\ \wedge\ \mathrm{rk}(\{A',B',C'\}) = 3\ \wedge\\
\mathrm{rk}(A,B,C,A',B',C') = 4 \Rightarrow\\
\mathrm{rk}(\{A,B,C,M\}) = 3\ \wedge\ \mathrm{rk}(\{A',B',C',M\}) = 3\ \wedge\\
\mathrm{rk}(\{A,B,C,N\}) = 3\ \wedge\ \mathrm{rk}(\{A',B',C',N\}) = 3\ \wedge\\
\mathrm{rk}(\{M,N\}) = 2\ \wedge\\
(\\
\mathrm{rk}(\{M,N,P\}) = 2 \Leftrightarrow\\
\mathrm{rk}(\{A,B,C,P\}) = 3\ \wedge\ \mathrm{rk}(\{A',B',C',P\}) = 3\\
)
\end{array}
$$

Utilization through a simple example (2)



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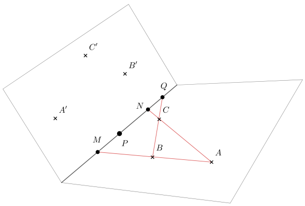

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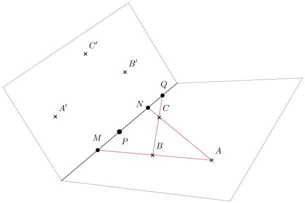
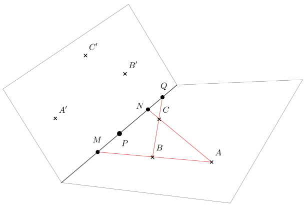


Lemma

With the previous notations, there is at least one point M in the intersection of the two planes. (Proof by A_{10}.)

Utilization through a simple example (2)



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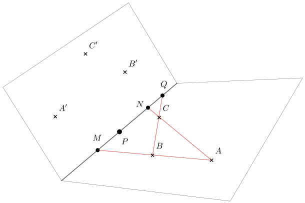
Lemma

In an incidence projective plane, if three points M, N and Q are on the three edges of a triangle ABC, then at least two of these three points are different.

Utilization through a simple example (2)



Lemma

In an incidence projective plane, if three points M, N and Q are on the three edges of a triangle ABC, then at least two of these three points are different.

There are two cases: $A = M$ or $A \neq M$:

Case $\text{rk}(\{A, M\}) = 2$.

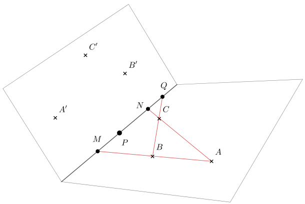
Then $\text{rk}(\{A, C, M, N, Q\}) = 3$ because

$\text{rk}(\{A, B, C, M, N, Q\}) + \text{rk}(\{A, M\}) \leq \text{rk}(\{A, B, M\}) + \text{rk}(\{A, C, M, N, Q\})$

with: $\text{rk}(\{A, C, M, N, Q\}) + \text{rk}(\{N\}) \leq \text{rk}(\{M, N, Q\}) + \text{rk}(\{A, C, N\})$

we have $\text{rk}(\{M, N, Q\}) \geq 2$.

Utilization through a simple example (2)



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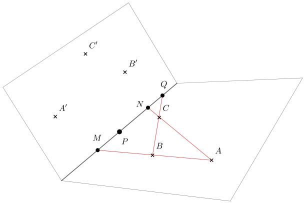
Lemma

In an incidence projective plane, if three points M, N and Q are on the three edges of a triangle ABC, then at least two of these three points are different.

Lemma

In a 3D incidence projective space, the intersection of two different planes is a line.

Utilization through a simple example (3)



$$
\begin{array}{l}
\forall\ A\ B\ C\ A'\ B'\ C'\ M\ N, P \\
\mathrm{rk}(\{A, B, C\}) = 3\ \wedge\ \mathrm{rk}(\{A', B', C'\}) = 3\ \wedge \\
\mathrm{rk}(A, B, C, A', B', C') = 4 \wedge \\
\mathrm{rk}(\{A, B, C, M\}) = 3\ \wedge\ \mathrm{rk}(\{A', B', C', M\}) = 3\ \wedge \\
\mathrm{rk}(\{A, B, C, N\}) = 3\ \wedge\ \mathrm{rk}(\{A', B', C', N\}) = 3\ \wedge \\
\mathrm{rk}(\{M, N\}) = 2\ \wedge \\
(\\
\mathrm{rk}(\{M, N, P\}) = 2 \Leftrightarrow \\
\mathrm{rk}(\{A, B, C, P\}) = 3\ \wedge\ \mathrm{rk}(\{A', B', C', P\}) = 3 \\
)
\end{array}
$$

Utilization through a simple example (3)

$\forall\ A\ B\ C\ A'\ B'\ C'\ M\ N, P$

$\mathrm{rk}(\{A, B, C\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C'\}) = 3\ \wedge$

$\mathrm{rk}(A, B, C, A', B', C') = 4\ \wedge$

$\mathrm{rk}(\{A, B, C, M\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', M\}) = 3\ \wedge$

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$\mathrm{rk}(\{A', B', C', N\}) = 3\ \wedge$

$\mathrm{rk}(\{M, N\}) = 2\ \wedge$

$\mathrm{rk}(\{A, B, C, P\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', P\}) = 3 \Rightarrow$

$\mathrm{rk}(\{M, N, P\}) = 2$

Utilization through a simple example (3)

$\forall\ A\ B\ C\ A'\ B'\ C'\ M\ N, P$

$rk(\{A, B, C\}) = 3\ \wedge$

$rk(\{A', B', C'\}) = 3\ \wedge$

$rk(A, B, C, A', B', C') = 4\ \wedge$

$rk(\{A, B, C, M\}) = 3\ \wedge$

$rk(\{A', B', C', M\}) = 3\ \wedge$

$rk(\{A, B, C, N\}) = 3\ \wedge$

$rk(\{A', B', C', N\}) = 3\ \wedge$

$rk(\{M, N\}) = 2\ \wedge$

$rk(\{A, B, C, P\}) = 3\ \wedge$

$rk(\{A', B', C', P\}) = 3 \Rightarrow$

$rk(\{M, N, P\}) = 2$

Sketch of a proof

$rk(\{A, B, C, M, P\}) = 3$

Utilization through a simple example (3)

$\forall\ A\ B\ C\ A'\ B'\ C'\ M\ N, P$

$\mathrm{rk}(\{A, B, C\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C'\}) = 3\ \wedge$

$\mathrm{rk}(A, B, C, A', B', C') = 4\ \wedge$

$\mathrm{rk}(\{A, B, C, M\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', M\}) = 3\ \wedge$

$\mathrm{rk}(\{A, B, C, N\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', N\}) = 3\ \wedge$

$\mathrm{rk}(\{M, N\}) = 2\ \wedge$

$\mathrm{rk}(\{A, B, C, P\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', P\}) = 3 \Rightarrow$

$\mathrm{rk}(\{M, N, P\}) = 2$

Sketch of a proof

$\mathrm{rk}(\{A, B, C, M, P\}) = 3$

$(i)\ \mathrm{rk}(\{A, B, C, M, P\}) \geq 3$

Utilization through a simple example (3)

$\forall\ A\ B\ C\ A'\ B'\ C'\ M\ N, P$

$\mathrm{rk}(\{A, B, C\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C'\}) = 3\ \wedge$

$\mathrm{rk}(A, B, C, A', B', C') = 4\ \wedge$

$\mathrm{rk}(\{A, B, C, M\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', M\}) = 3\ \wedge$

$\mathrm{rk}(\{A, B, C, N\}) = 3\ \wedge$

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$\mathrm{rk}(\{M, N\}) = 2\ \wedge$

$\mathrm{rk}(\{A, B, C, P\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', P\}) = 3 \Rightarrow$

$\mathrm{rk}(\{M, N, P\}) = 2$

Sketch of a proof

$\mathrm{rk}(\{A, B, C, M, P\}) = 3$

$(i)\ \mathrm{rk}(\{A, B, C, M, P\}) \geq 3$

$$(ii)\ \mathrm{rk}(\{A, B, C, M, P\}) + \mathrm{rk}(\{A, B, C\}) \leq \mathrm{rk}(\{A, B, C, M\}) + \mathrm{rk}(\{A, B, C, P\})$$

Utilization through a simple example (3)

$\forall\ A\ B\ C\ A'\ B'\ C'\ M\ N, P$

$\mathrm{rk}(\{A, B, C\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C'\}) = 3\ \wedge$

$\mathrm{rk}(A, B, C, A', B', C') = 4\ \wedge$

$\mathrm{rk}(\{A, B, C, M\}) = 3\ \wedge$

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$\mathrm{rk}(\{A, B, C, N\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', N\}) = 3\ \wedge$

$\mathrm{rk}(\{M, N\}) = 2\ \wedge$

$\mathrm{rk}(\{A, B, C, P\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', P\}) = 3 \Rightarrow$

$\mathrm{rk}(\{M, N, P\}) = 2$

Sketch of a proof

$\mathrm{rk}(\{A, B, C, M, P\}) = 3$

$(i)\ \mathrm{rk}(\{A, B, C, M, P\}) \geq 3$

$(ii)\ \mathrm{rk}(\{A, B, C, M, P\}) + 3$

$$\leq$$

$$3 + 3$$

Utilization through a simple example (3)

$\forall\ A\ B\ C\ A'\ B'\ C'\ M\ N, P$

$\mathrm{rk}(\{A, B, C\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C'\}) = 3\ \wedge$

$\mathrm{rk}(A, B, C, A', B', C') = 4\ \wedge$

$\mathrm{rk}(\{A, B, C, M\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', M\}) = 3\ \wedge$

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$\mathrm{rk}(\{A, B, C, P\}) = 3\ \wedge$

$\mathrm{rk}(\{A', B', C', P\}) = 3 \Rightarrow$

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Sketch of a proof

$\mathrm{rk}(\{A, B, C, M, P\}) = 3$

$\mathrm{rk}(\{A', B', C', M, P\}) = 3$

$\mathrm{rk}(\{A, B, C, M, N, P\}) = 3$

$\mathrm{rk}(\{A', B', C', M, N, P\}) = 3$

$\mathrm{rk}(\{A, B, C, A', B', C', M, N, P\}) = 4$

$\mathrm{rk}(\{M, N, P\}) = 2$

The incomplete matroid problem

Problems

Let E be a given set and r be a rank function on E. The value of r is only known for some subsets of E,

- is r fully defined on $\mathcal{P}(E)$?
- given a set $A \subseteq E$, is it possible to compute $r(A)$?

It is possible to answer by using a simple but tedious method.

Rules of the game

Considering the powerset $\mathcal{P}(E)$ where the bounds of the rank function are initialized for each set, one applies as much as possible the following rules.
8 rules corresponding to axioms A_2 and A_3:

Monotonicity

(r_1) **if** $X \subseteq Y$ and $rkMin(X) > rkMin(Y)$ **then** $rkMin(Y) \leftarrow rkMin(X)$
(r_2) **if** $Y \subseteq X$ and $rkMin(Y) > rkMin(X)$ **then** $rkMin(X) \leftarrow rkMin(Y)$
(r_3) **if** $X \subseteq Y$ and $rkMax(Y) < rkMax(X)$ **then** $rkMax(X) \leftarrow rkMax(Y)$
(r_4) **if** $Y \subseteq X$ and $rkMax(X) < rkMax(Y)$ **then** $rkMax(Y) \leftarrow rkMax(X)$

Submodularity

(r_5) **if** $rkMax(X) + rkMax(Y) - rkMin(X \cap Y) < rkMax(X \cup Y)$
then $rkMax(X \cup Y) \leftarrow (rkMax(X) + rkMax(Y) - rkMin(X \cap Y))$
(r_6) **if** $rkMax(X) + rkMax(Y) - rkMin(X \cup Y) < rkMax(X \cap Y)$
then $rkMax(X \cap Y) \leftarrow (rkMax(X) + rkMax(Y) - rkMin(X \cup Y))$
(r_7) **if** $rkMin(X \cap Y) + rkMin(X \cup Y) - rkMax(Y) > rkMin(X)$
then $rkMin(X) \leftarrow (rkMin(X \cap Y) + rkMin(X \cup Y) - rkMax(Y))$
(r_8) **if** $rkMin(X \cap Y) + rkMin(X \cup Y) - rkMax(X) > rkMin(Y)$
then $rkMin(Y) \leftarrow (rkMin(X \cap Y) + rkMin(X \cup Y) - rkMax(X))$

Implementation

A matroid Based Incidence geometry Prover (Bip)

- basic solver of the incomplete matroid problem in the geometric case ... but yielding proofs that can be automatically verified by Coq;
- originally two versions: 2D and 3D;
- aimed to help a mathematician in proving “small parts” of a theorem ... but used to prove significant theorems;
- huge complexity (at least exponential in the number of points)

Limitations

A closed world hypothesis

The previous method works on a given set of points E.
Theoretically, it is possible to add some auxiliary point on the fly, but we then face to a huge complexity.

Disjunctive situations

Sometimes, there are several possibilities, but to complete each of them the different cases have to be explicitly given by the user (see the example above).

Usability

In the first prototype, all was hard codded and a re-compilation was needed for each example.
Poor interaction with Coq.
$\Rightarrow$ IO with files and *ad hoc* description language

Limitations

Complexity and huge Coq proofs

As said before, the time and space complexities are exponential, but also the proofs can be huge (several dozen of kilo-lines). Coq is unable to treat a monolithic of that size.

⇒ systematic decomposition into basic lemmas.

⇒ several conclusions taken into account.

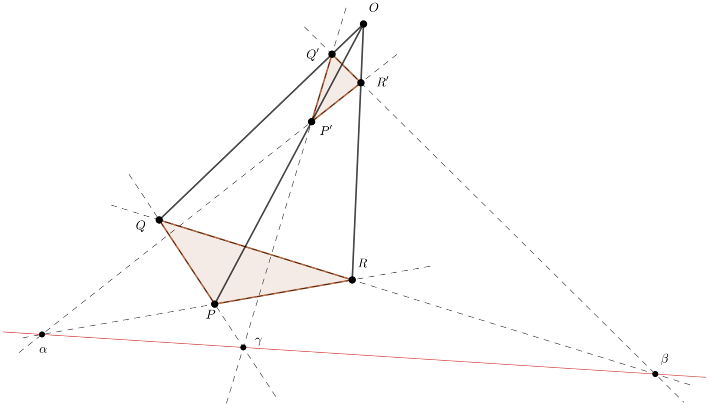
Dimensions

The initial prototype only dealt with dimensions 2 and 3.

⇒ small changes on data structures and small changes in Coq context to deal with higher dimensions.

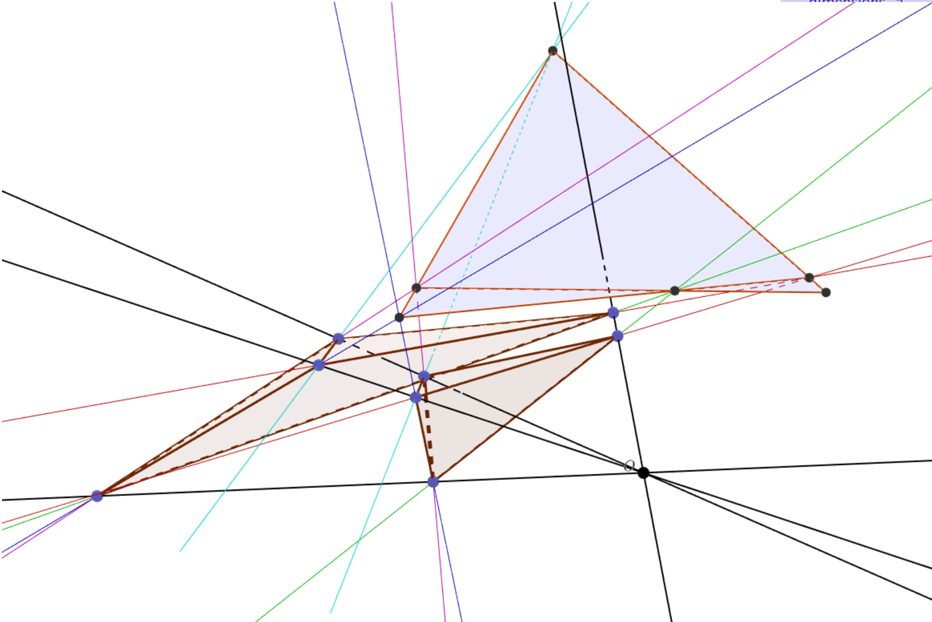
Desargues's theorem in 3D and 4D

Recall : a crucial step in the proof of Desargues's theorem in 2D, sometimes called 2.5d configuration.



Desargues's theorem in 3D and 4D

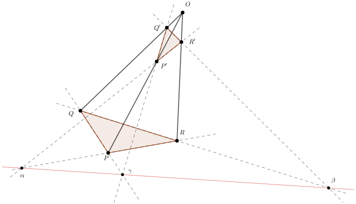
Real 3D configuration (figure made with Geogebra 3D):



Desargues's theorem in 3D and 4D

Desargues's theorem in 4D

given 2 pentachores P and P' which are in perspective from a point O, the 10 points defined by the intersection of the corresponding edges define a 3D space and form a 2.5d Desargues's configuration.



Desargues's theorem in 3D and 4D

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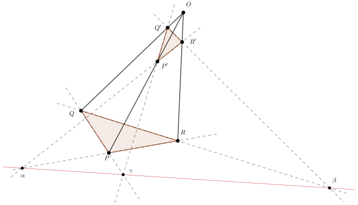
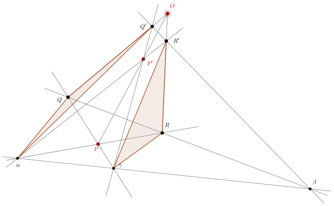
Symmetry





Desargues's theorem in 3D and 4D

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context
  dimension 4
endofcontext
 points
   Oo A B C D E Ap Bp Cp Dp Ep
   ab ac ad  ae bc bd be cd ce de
 hypotheses
   A B C D E : 5
   A Ap : 2
   B Bp : 2
   C Cp : 2
   D Dp : 2
   E Ep : 2
   Ap Bp Cp Dp Ep : 5
   Oo A Ap : 2
   Oo B Bp : 2      # B
   Oo C Cp : 2      # C
   Oo D Dp : 2      # D
   Oo E Ep : 2      # E
   ab A B : 2
   ab Ap Bp : 2
   ac A C : 2
   ac Ap Cp : 2
   ad A D : 2
   ad Ap Dp : 2
   ae A E : 2
   ae Ap Ep : 2
   bc B C : 2
   bc Bp Cp : 2
   bd B D : 2
   bd Bp Dp : 2
   be B E : 2
   be Bp Ep : 2
   cd C D : 2
   cd Cp Dp : 2
   ce C E : 2
   ce Cp Ep : 2
   de D E : 2
   de Dp Ep : 2
   Oo B C D E : 5
   A Oo C D E : 5
   A B Oo D E : 5
   A B C Oo E : 5
   A B C D Oo : 5
   Oo Bp Cp Dp Ep : 5
   Ap Oo Cp Dp Ep : 5
   Ap Bp Oo Dp Ep : 5
   Ap Bp Cp Oo Ep : 5
   Ap Bp Cp Dp Oo : 5
 conclusion
   ab ac ad ae bc bd be cd ce de : 4
   cd ce de : 2
   bd be de : 2
   bc be ce : 2
   bc bd cd : 2
   ad ae de : 2
   ac ae ce : 2
   ac ad cd : 2
   ab ae be : 2
   ab ad bd : 2
   ad ac bc : 2
end
```

Desargues's theorem in 3D and 4D

4D

- 21 points involved
- the proof of 11 values of ranks are required
- Computation time : about 1 week
- Coq file size : 47,4 Mb
- number of lines : 497157 (a lot of comments)
- number of lemmas : 2517

For the first line of the conclusion

- Computation time : about 1 week
- Coq file size : 6.2 Mb (x 11 = 68.2 Mb)
- number of lines : 62,000 (x 11 = 682,000 lines)
- number of lemmas : 635 (x 11 = 6985)

Desargues's theorem in 5D

Theorem

In a projective incidence space of dimension 5, for all couple of 5-simplexes which are in perspective from a point O if the 15 couples of corresponding edges intersect each in exactly one point, then

- these points belong to a 4-dimensional space H, and
- they form a figure composed by the vertices of a pentachore P and the intersection of the edges of P with a hyperplane of H.

Comments

- 28 points are involved,
- the expected computation time will be about 128 weeks $\sim$ 2 years and a half (with my old PC)

Simple example in 5D

Hyperplanes in 5D

- (Axiom) In 5D, the intersection of a hyperplane (dim 4) and a line is at least a point:
  $\forall\ A\ B\ C\ D\ E\ M\ N,\ exists\ P,$
  $\mathrm{rk}(\{A, B, C, D, E\}) = 5 \wedge \mathrm{rk}(\{M, N\}) = 2 \Rightarrow$
  $\mathrm{rk}(\{A, B, C, D, E, P\}) = 5 \wedge \mathrm{rk}(\{M, N, P\}) = 2$

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- (Theorem) In 5D, the intersection of two distinct hyperplanes is a 3-dimensional space.
  Sketch of the proof:
  - there are four independent points in the intersection (5 cases)
  - these four points span the intersection (double inclusion)

Existence (none of A ... E is in the intersection)

```
# there are 4 points in <ABCDE> inter <A'B'C'D'E'> case ABCDE notin <A'B'C'D'E'>
context
    dimension 5
    layers 1
endofcontext
layer 0
  points
    A B C D E A' B' C' D' E' p1 p2 p3 p4
hypotheses
    A B C D E : 5
    A' B' C' D' E' : 5
    A B C D E A' B' C' D' E' : 6
    p1 A B : 2
    p1 A : 2
    p1 B : 2
    p2 A C : 2
    p2 C : 2
    p2 A : 1
    p3 A D : 2
    p3 D : 2
    p3 A : 2
    p4 A E : 2
    p4 A : 2
    p4 E : 2
    p1 A' B' C' D' E' : 5
    p2 A' B' C' D' E' : 5
    p3 A' B' C' D' E' : 5
    p4 A' B' C' D' E' : 5
  conclusion
    None
endoflayer
  conclusion
    p1 p2 p3 p4 : 4
end
```

Simple example in 5D

```
# in dim 5, the intersection of 2 different 4-dimensional space is 3-dimensional
# all subsets of 5 points (as independent as possible) have a rank equal to 4
context
    dimension 5
    layers 1
endofcontext
layer 0
  points
    A B C D E A' B' C' D' E' I J K L M
  hypotheses
    A B C D E : 5
    A' B' C' D' E' : 5
    A B C D E A' B' C' D' E' : 6
    I A B C D E : 5
    J A B C D E : 5
    K A B C D E : 5
    L A B C D E : 5
    M A B C D E : 5
    I A' B' C' D' E' : 5
    J A' B' C' D' E' : 5
    K A' B' C' D' E' : 5
    L A' B' C' D' E' : 5
    M A' B' C' D' E' : 5
    I J K L : 4
  conclusion
    None
endoflayer
  conclusion
    I J K L M : 4
end
```

Simple example in 5D

```
# in dim 5, the intersection of 2 different 4-dimensional space is 3-dimensional
# here : the 3-space is included in the intersection.
context
    dimension 5
    layers 1
endofcontext
layer 0
  points
    A B C D E A' B' C' D' E' I J K L M
  hypotheses
    A B C D E : 5
    A' B' C' D' E' : 5
    A B C D E A' B' C' D' E' : 6
    I A B C D E : 5
    J A B C D E : 5
    K A B C D E : 5
    L A B C D E : 5
    I A' B' C' D' E' : 5
    J A' B' C' D' E' : 5
    K A' B' C' D' E' : 5
    L A' B' C' D' E' : 5
    I J K L : 4
    I J K L M : 4
  conclusion
    None
endoflayer
  conclusion
    M A B C D E : 5
    M A' B' C' D' E' : 5
end
```

Conclusion

A positive conclusion

The matroid approach easily allows to consider incidence geometry in higher dimensions.

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The matroid approach easily allows to consider incidence geometry in higher dimensions.

However

- it suffers of a huge complexity
- it is not fully automatized:
  - existential quantification not taken into account
  - incapacity to deal with several cases

Conclusion

A positive conclusion

The matroid approach easily allows to consider incidence geometry in higher dimensions.

However

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- it is not fully automatized:
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To be continued

- interactivity
- consider smarter algorithms