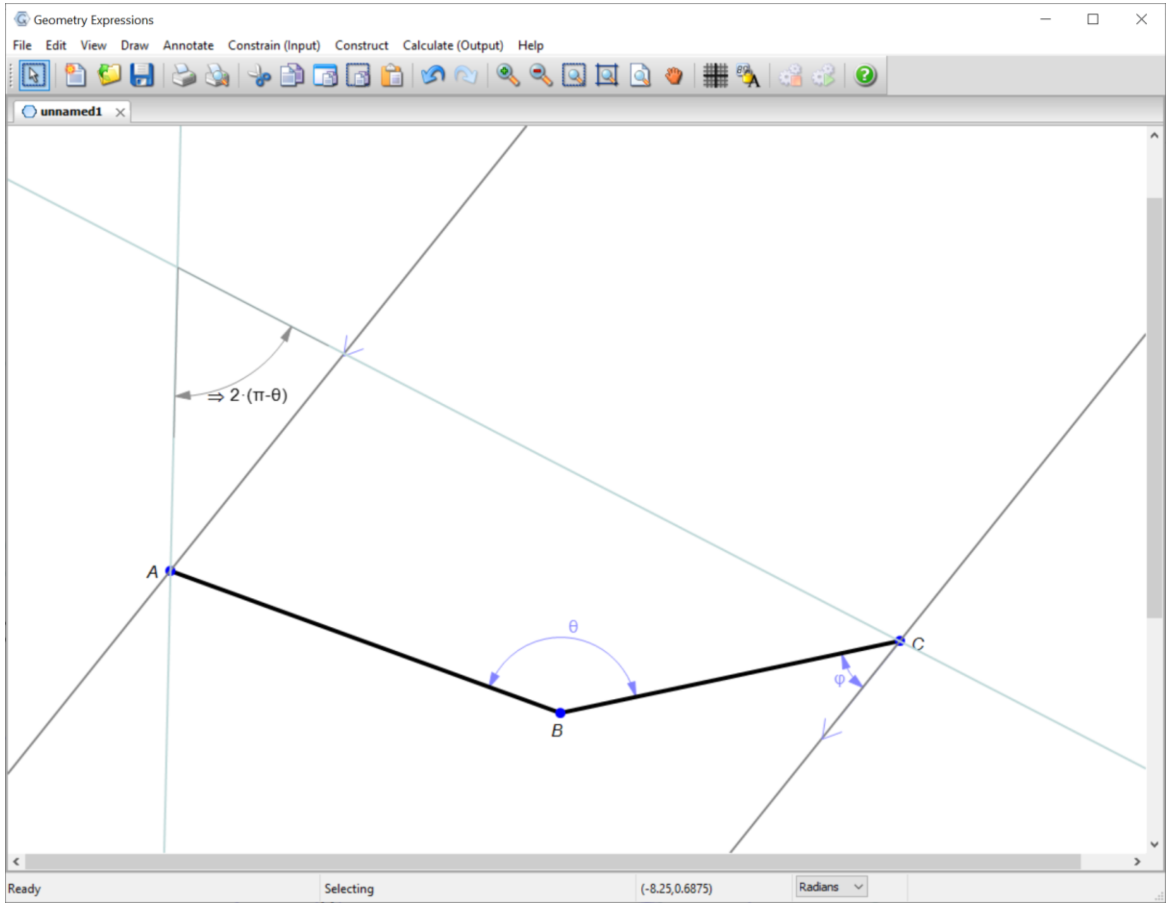
A method for the automated discovery of angle theorems

Philip Todd

Saltire Software

philt@saltire.com

Geometry Expressions Angle Engine using the Naïve Angle Method



Reviewer suggested:

Would like to see how many problems this addresses:

Ideally a list of hundreds of theorems

Naïve Angle Method

angle between line i and j is φ : $d_i - d_j = \varphi$

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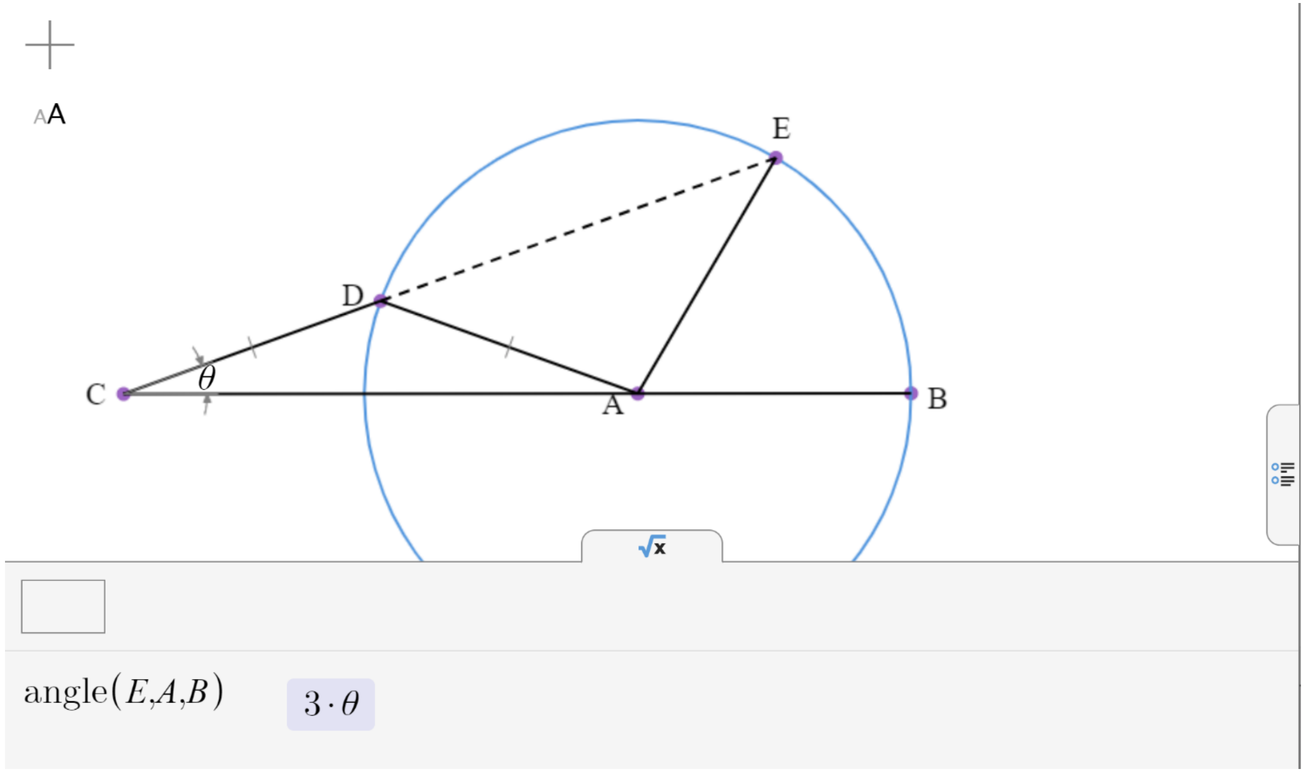
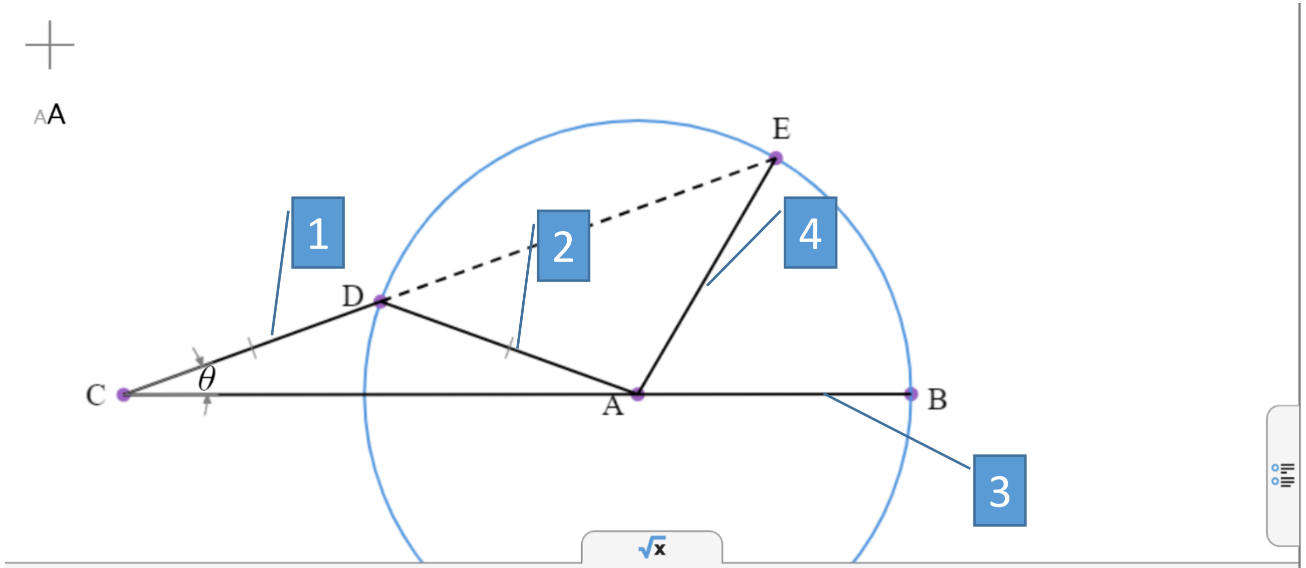
line k is the base of an isosceles triangle whose equal sides are i and j : $2\ d_k - d_i - d_j = \pi$

line j is the image of line i under reflection in k : $2\ d_k - d_i - d_j = 0$

from Archimedes' Liber Assumptorum

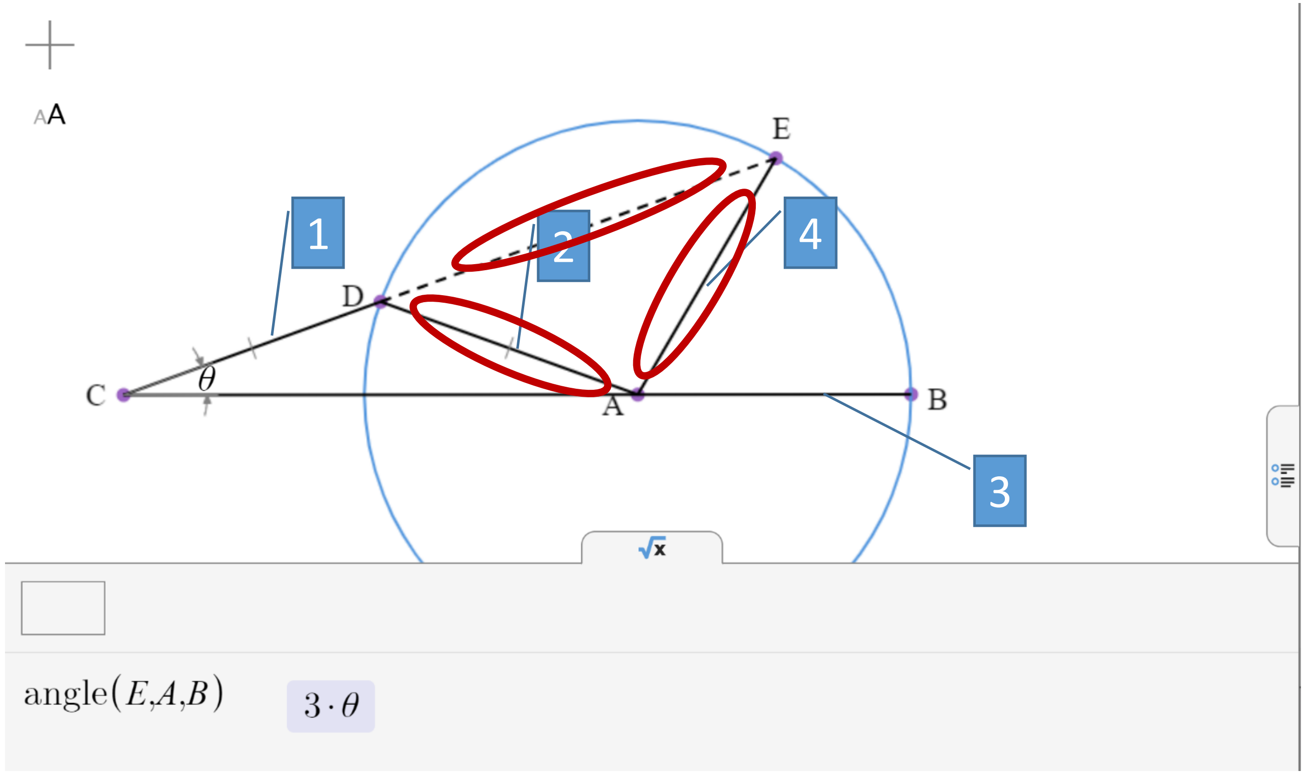


$$\begin{pmatrix} 2 & -1 & 0 & -1 & \pi \\ -1 & -1 & 2 & 0 & \pi \\ 1 & 0 & -1 & 0 & \theta \end{pmatrix}$$

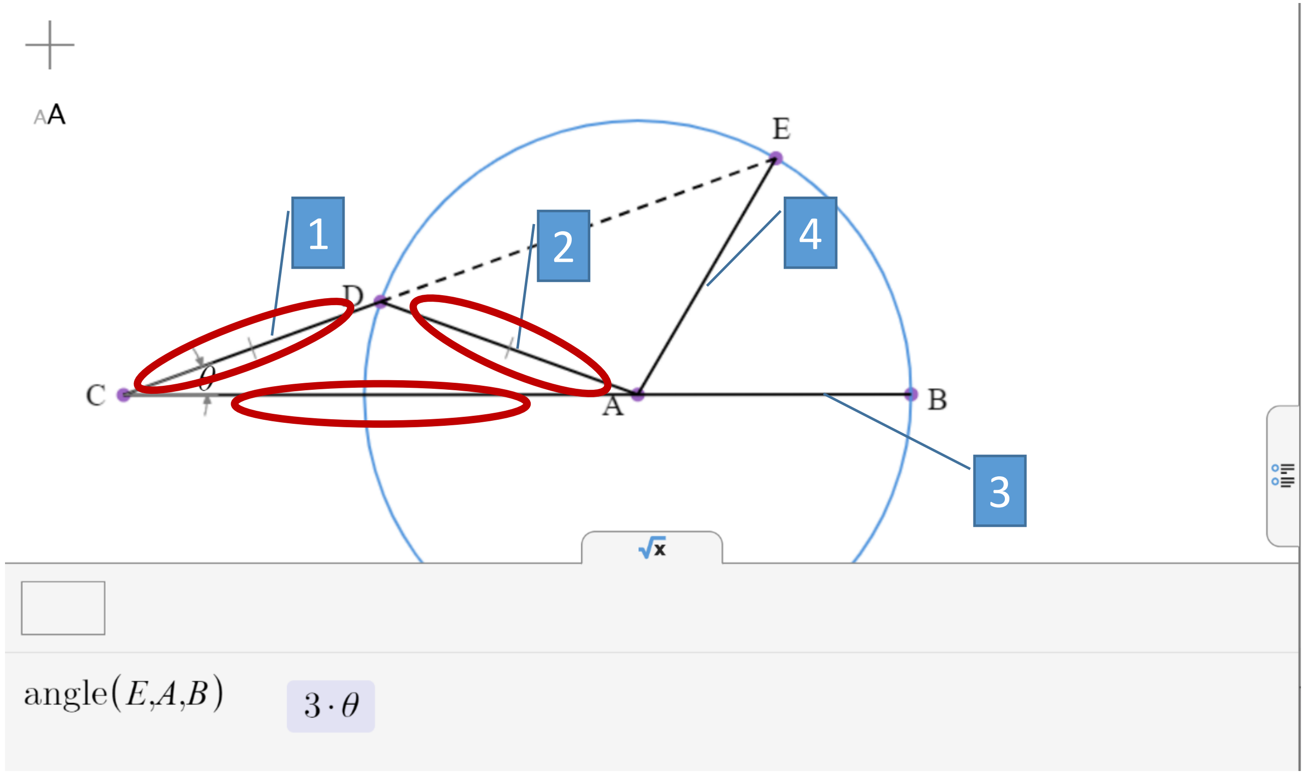


angle(E,A,B) $3\cdot\theta$

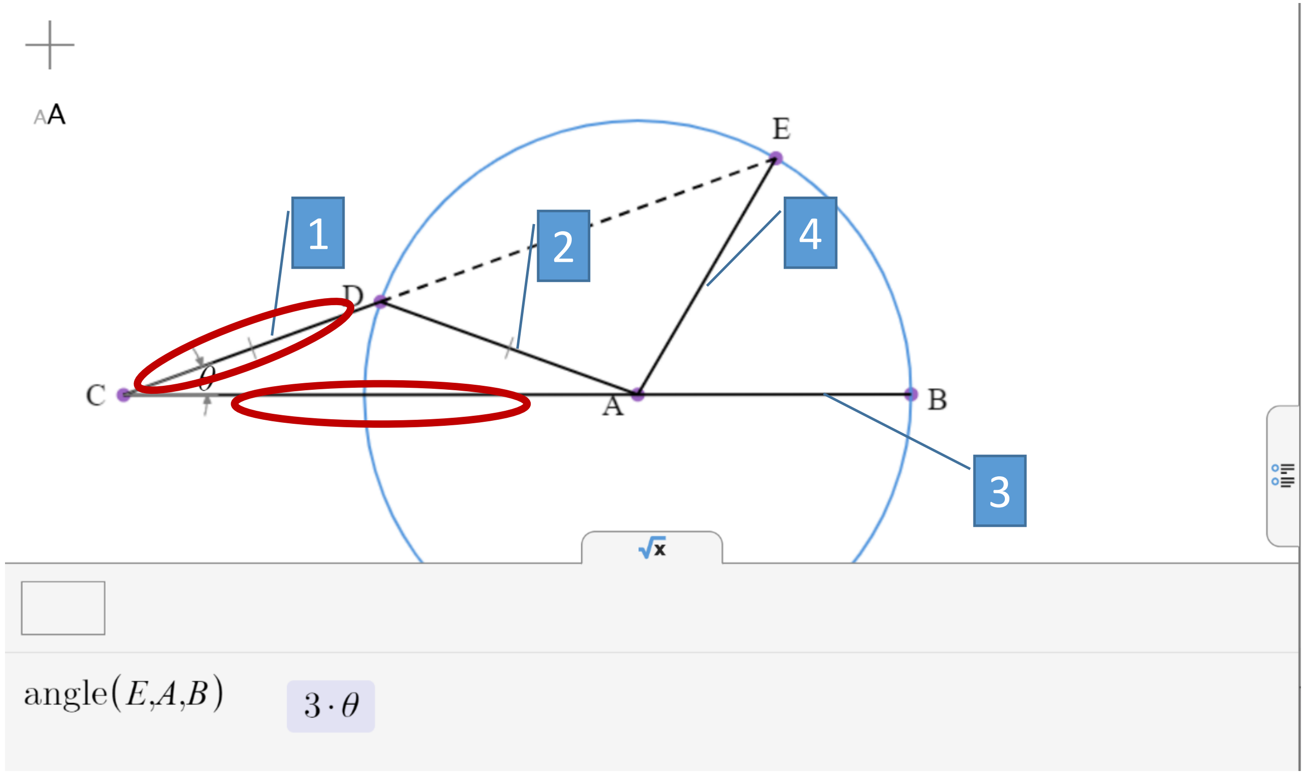
$$\begin{pmatrix} 2 & -1 & 0 & -1 & \pi \\ -1 & -1 & 2 & 0 & \pi \\ 1 & 0 & -1 & 0 & \theta \end{pmatrix}$$

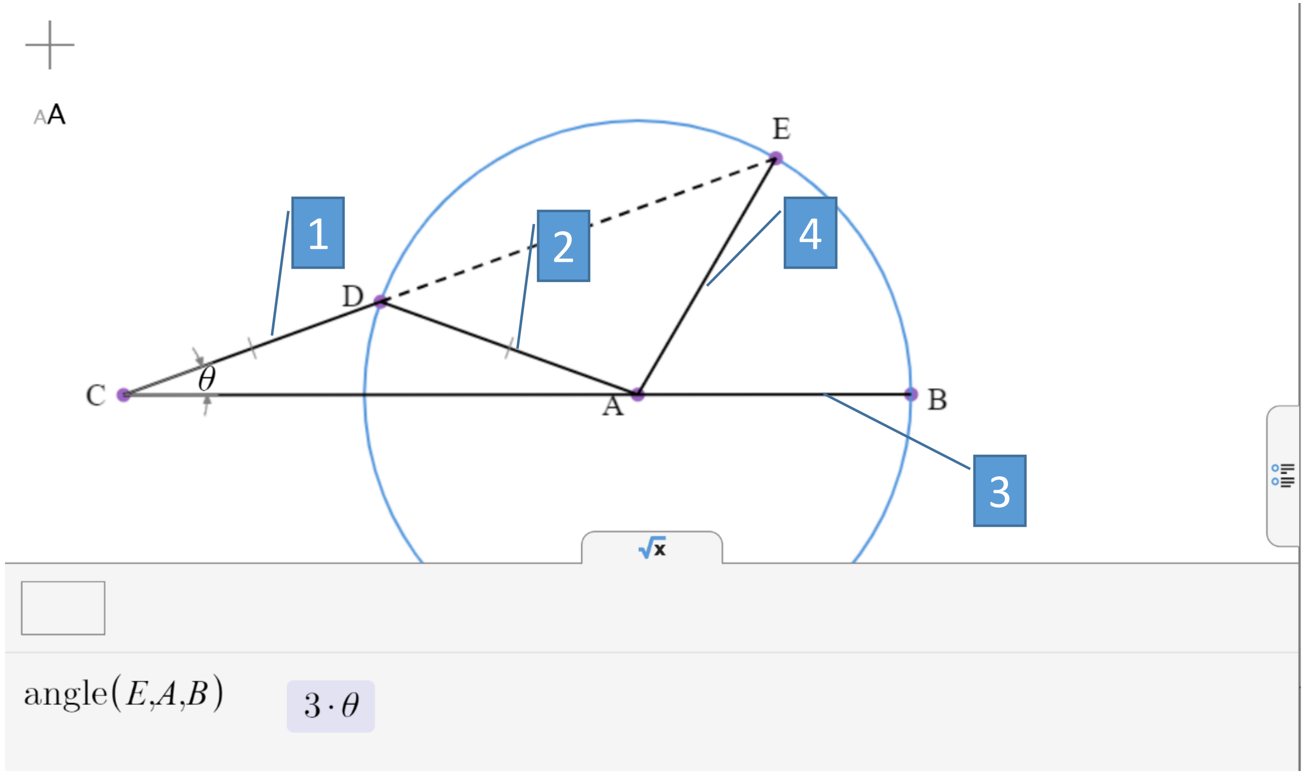


$$\begin{pmatrix} 2 & -1 & 0 & -1 & \pi \\ -1 & -1 & 2 & 0 & \pi \\ 1 & 0 & -1 & 0 & \theta \end{pmatrix}$$



$$\begin{pmatrix} 2 & -1 & 0 & -1 & \pi \\ -1 & -1 & 2 & 0 & \pi \\ 1 & 0 & -1 & 0 & \theta \end{pmatrix}$$





$$\begin{pmatrix} 2 & -1 & 0 & -1 & \pi \\ -1 & -1 & 2 & 0 & \pi \\ 1 & 0 & -1 & 0 & \theta \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 & -1 & \pi \\ 0 & -3 & 4 & -1 & 3\pi \\ 0 & -1 & 2 & -1 & \pi - 2\theta \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 & -1 & \pi \\ 0 & -3 & 4 & -1 & 3\pi \\ 0 & 0 & -2 & 2 & 6\theta \end{pmatrix}$$

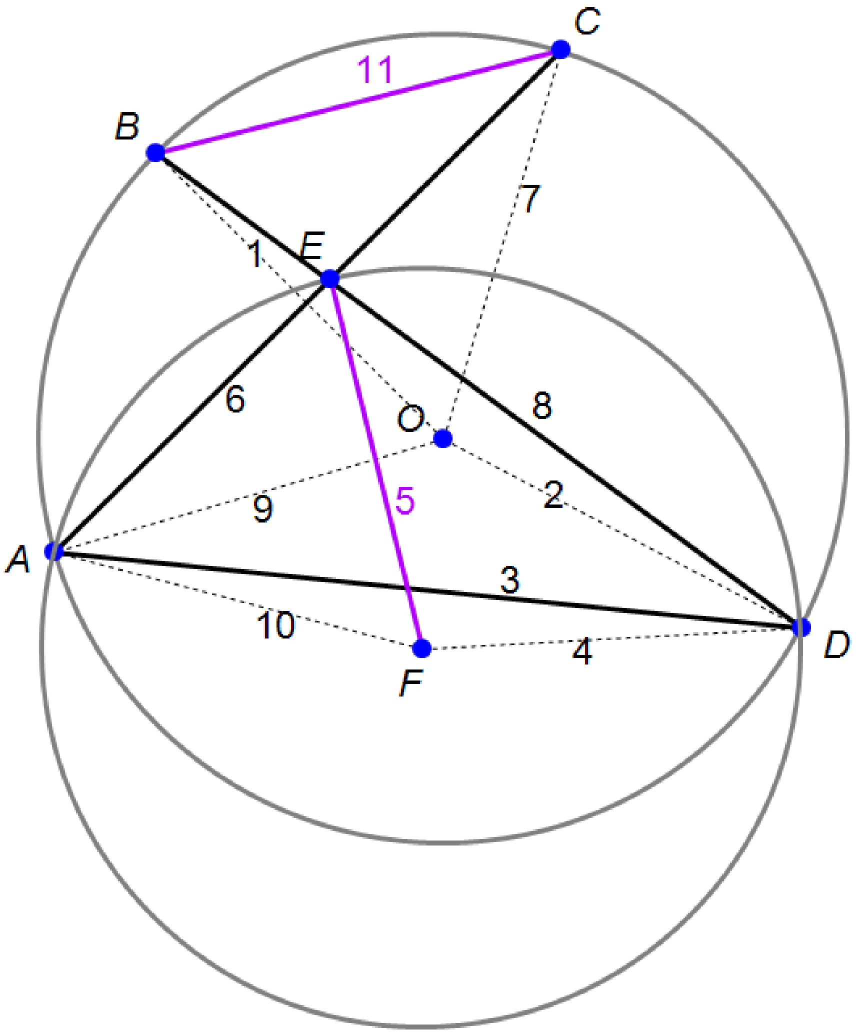
$$2(d_4 - d_3) = 6\theta$$

$$(d_4 - d_3) = 3\theta$$

Plan for “hundreds of theorems”

- Find some good theorems in books
- Extract their matrix
- Find similar theorem-bearing matrices
- Develop methods of deriving geometry theorems from these matrices

Geometry->Matrix

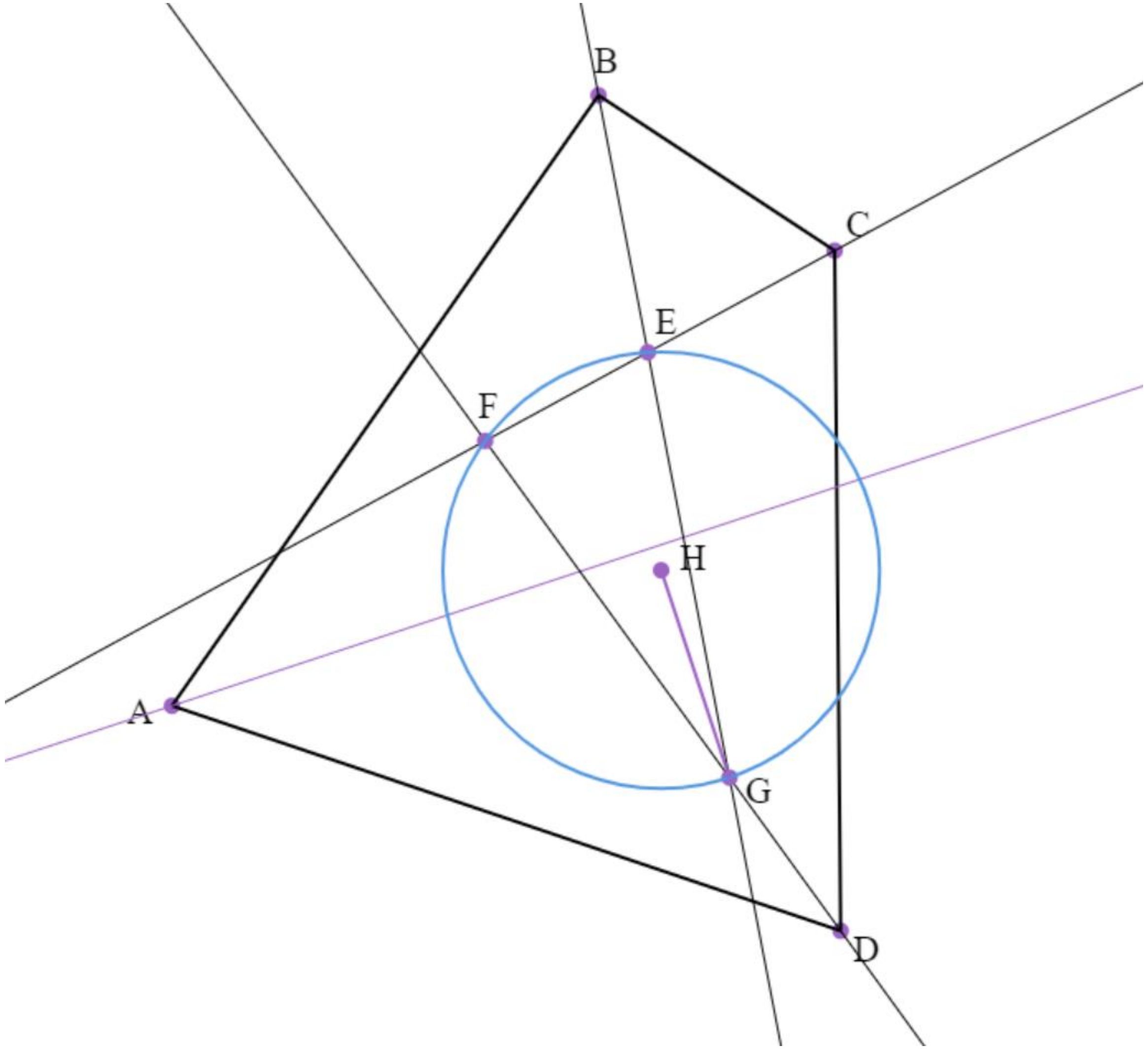


$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 & 0 & 0
\end{pmatrix}$$

BC is perpendicular to EF

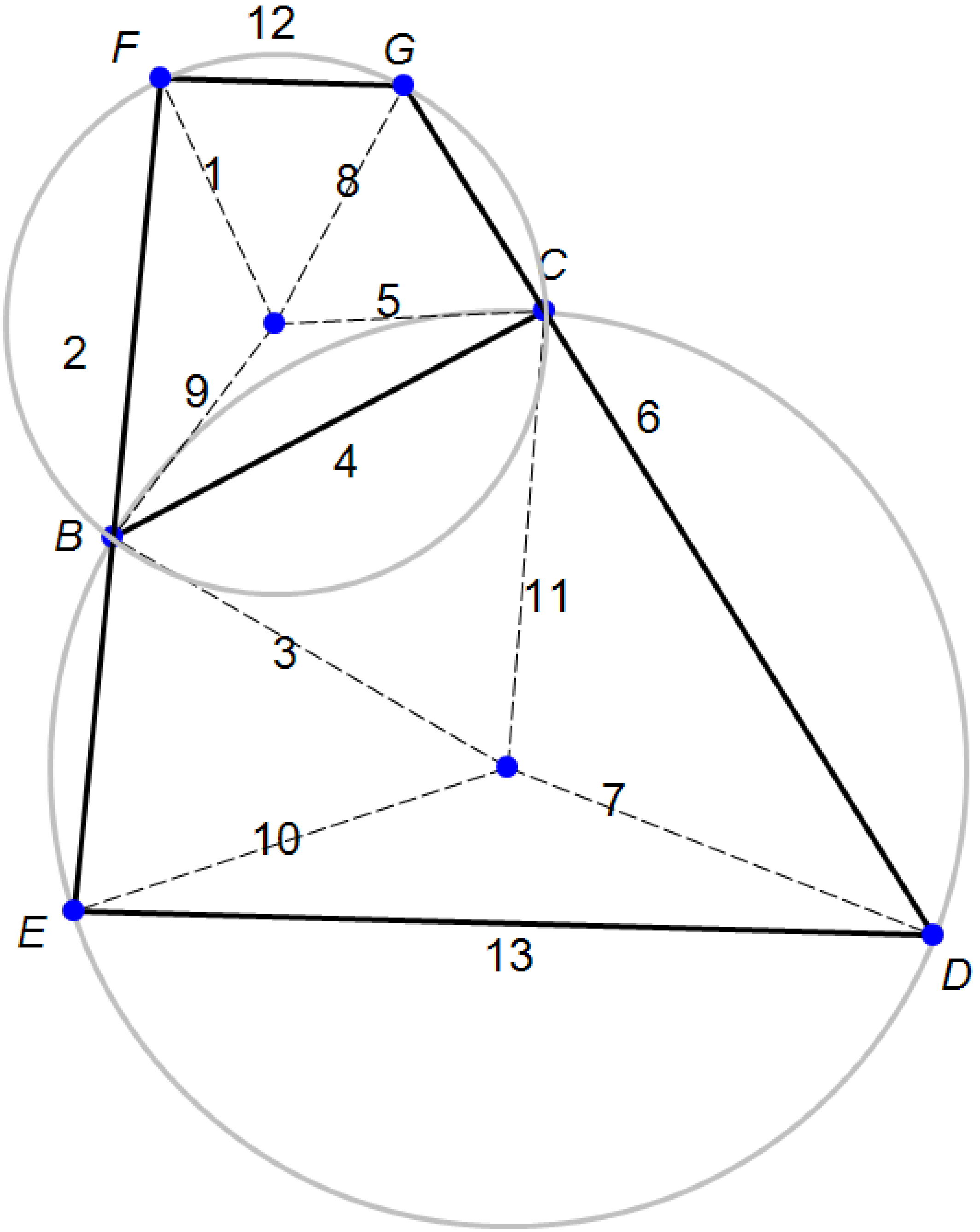
Chou, Gao & Zhang, “Automated Deduction in Geometry”

Geometry->Matrix->Geometry



$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

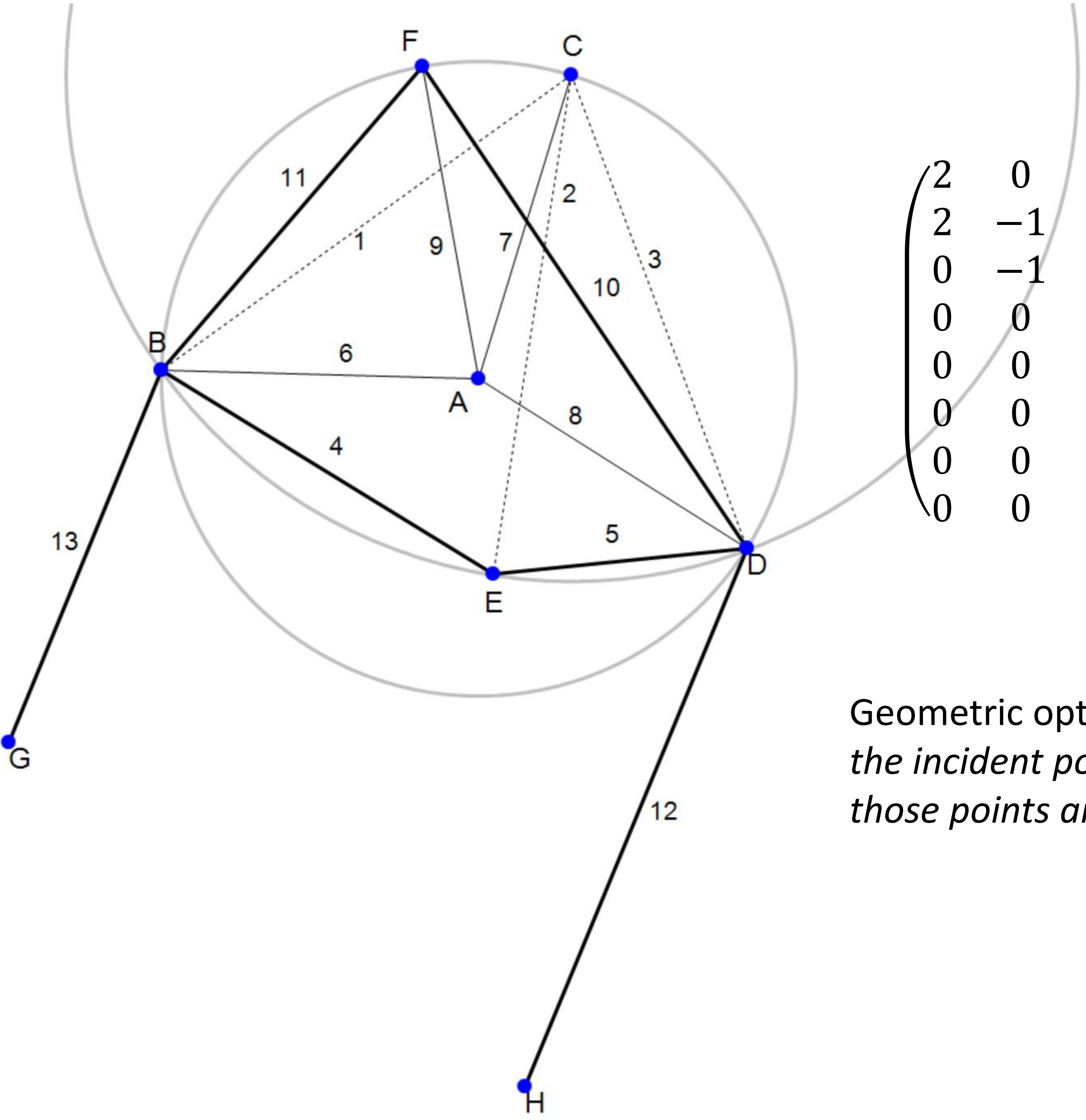
GH is perpendicular to the angle bisector of BAD



$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

FG is parallel to DE

Chou, Gao & Zhang, “Automated Deduction in Geometry”



$$
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1
\end{pmatrix}
$$

Geometric optics - *Reflected parallel rays meet on the circumcircle of the incident points and the circumcenter of the triangle formed by those points and the intersection of the mirror tangents.*

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}$$

$$\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1
\end{pmatrix}$$

Matrix Shape

Definition: We define the ***Shape Hypergraph*** of a matrix M to be the hypergraph with vertices corresponding to the rows of M, hyperedges corresponding to the columns of M and whose incidence matrix has zero and non-zero elements in the same positions as M

Matrix

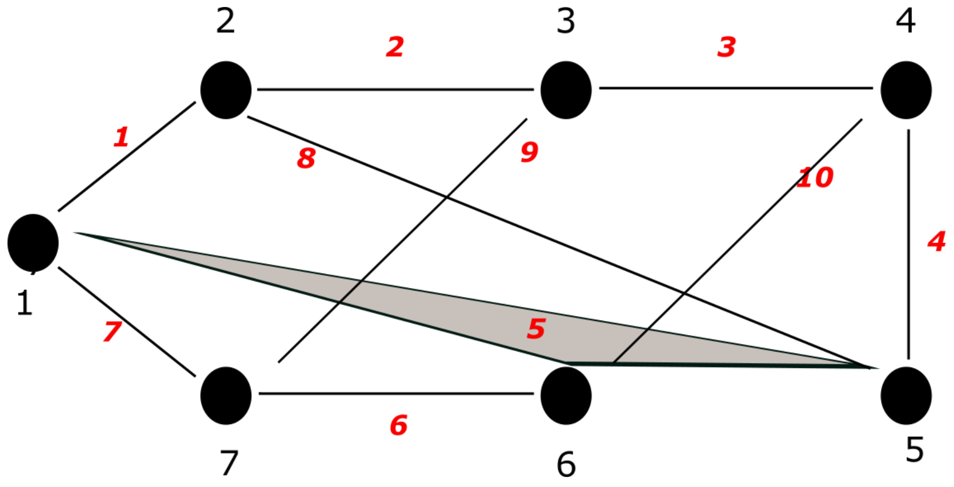
$$\begin{pmatrix} -1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 & 0 \end{pmatrix}$$

Incidence Matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

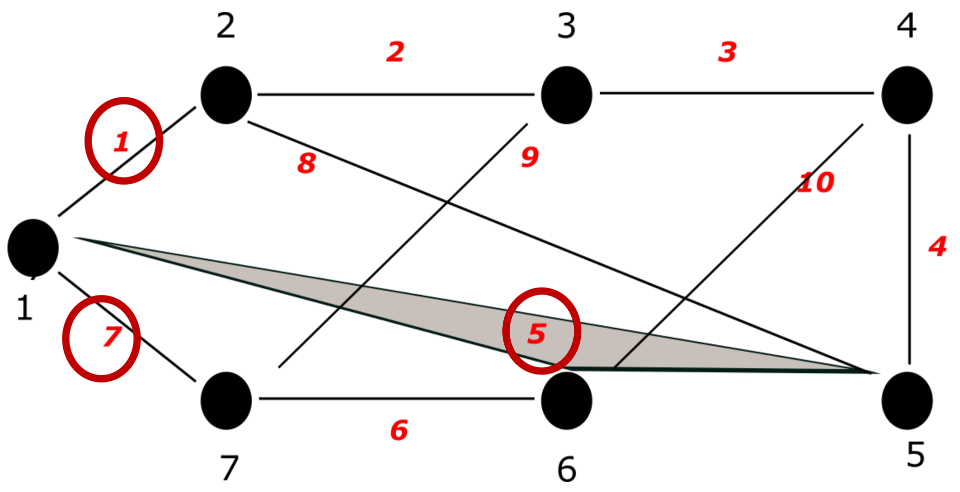
Shape Hypergraph

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$



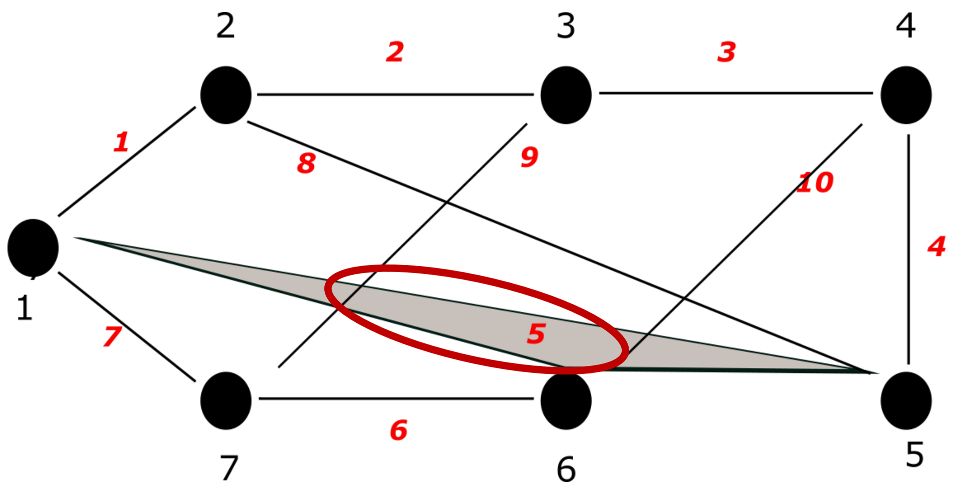
Shape Hypergraph

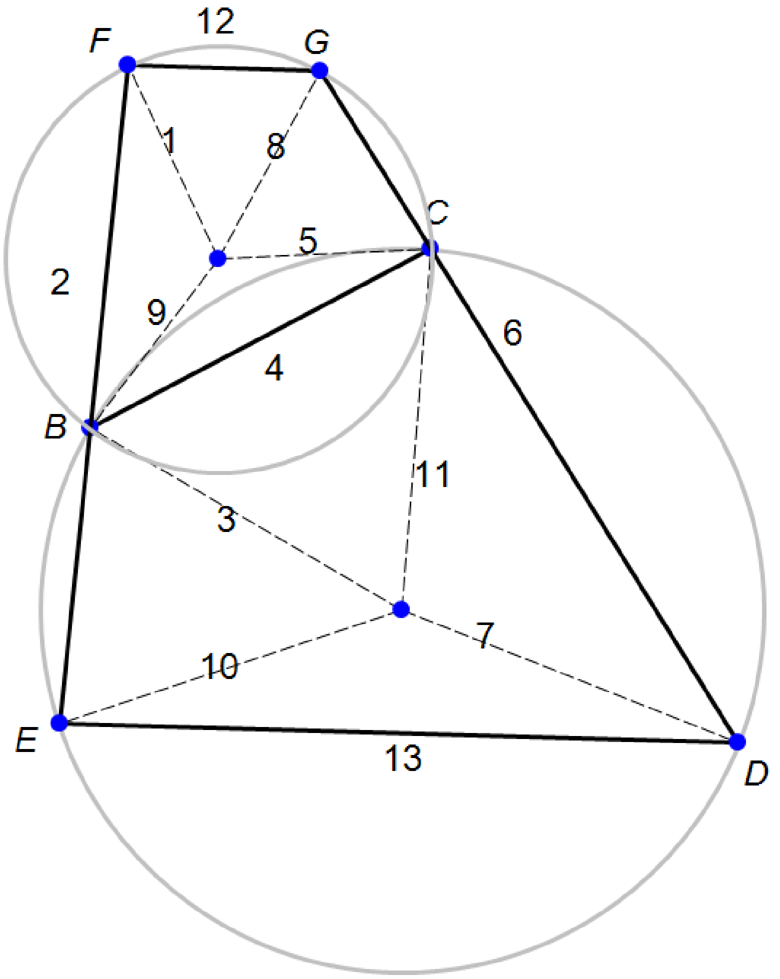
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$



Shape Hypergraph

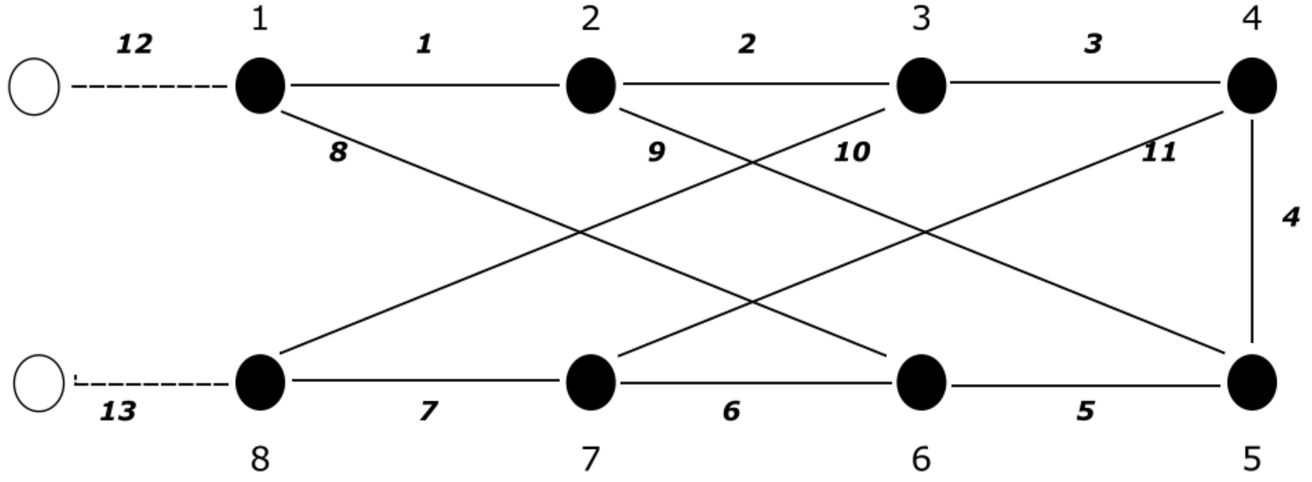
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

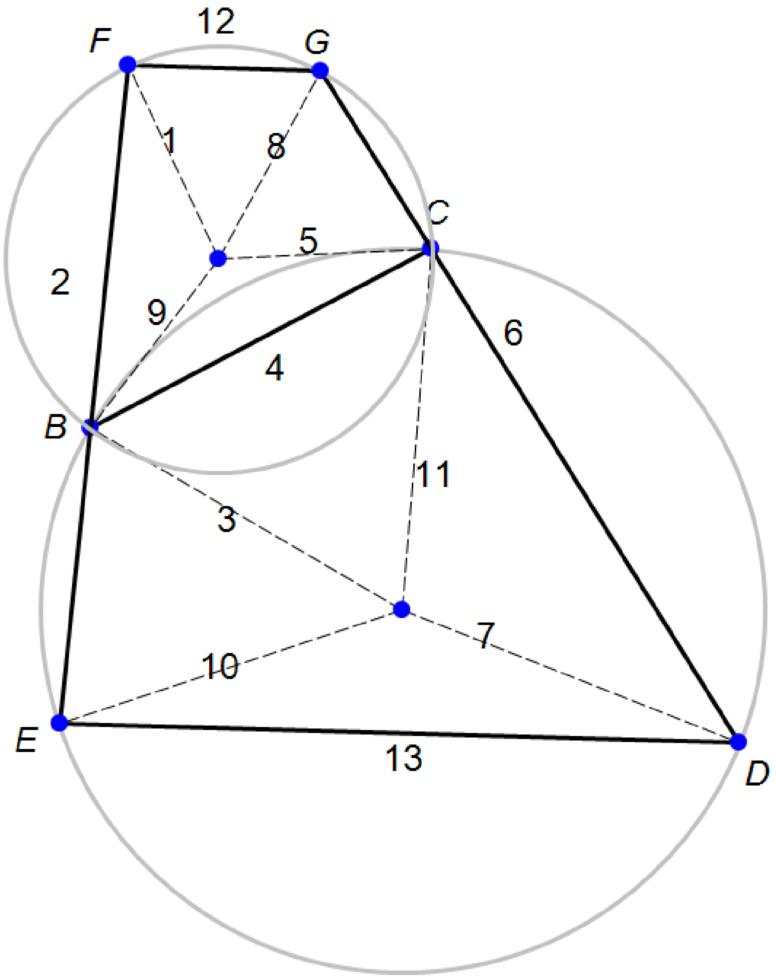




$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

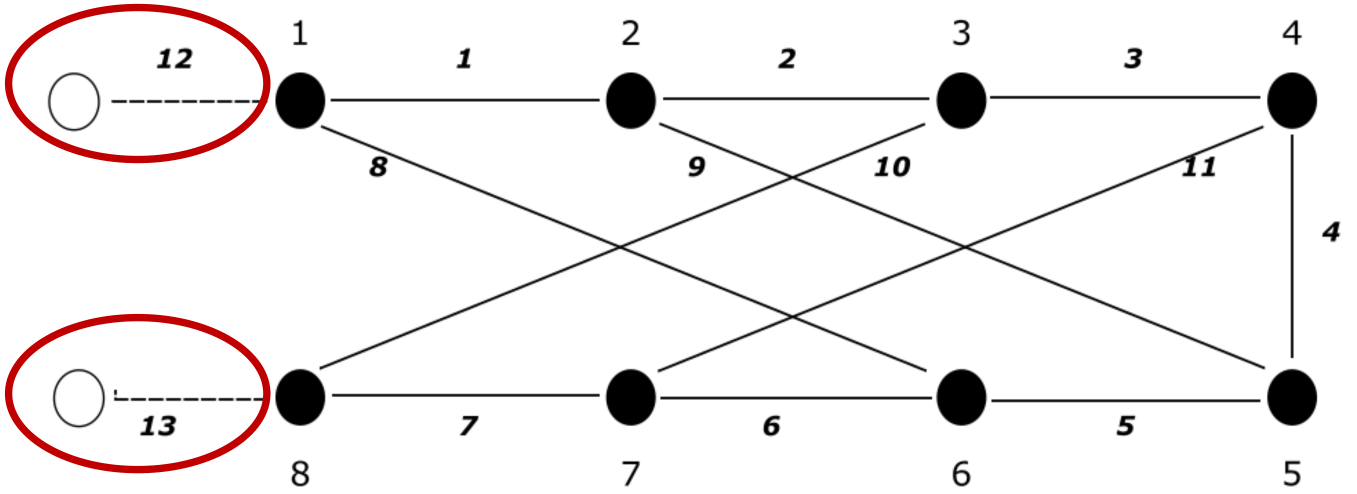
FG is parallel to DE

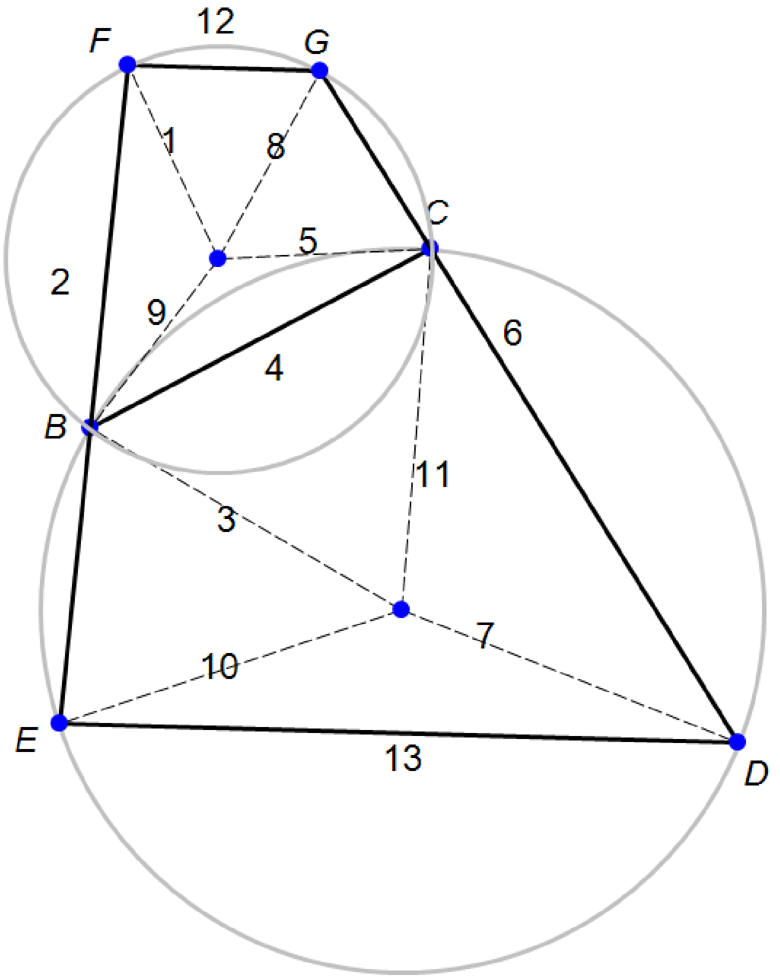




$$
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}
$$

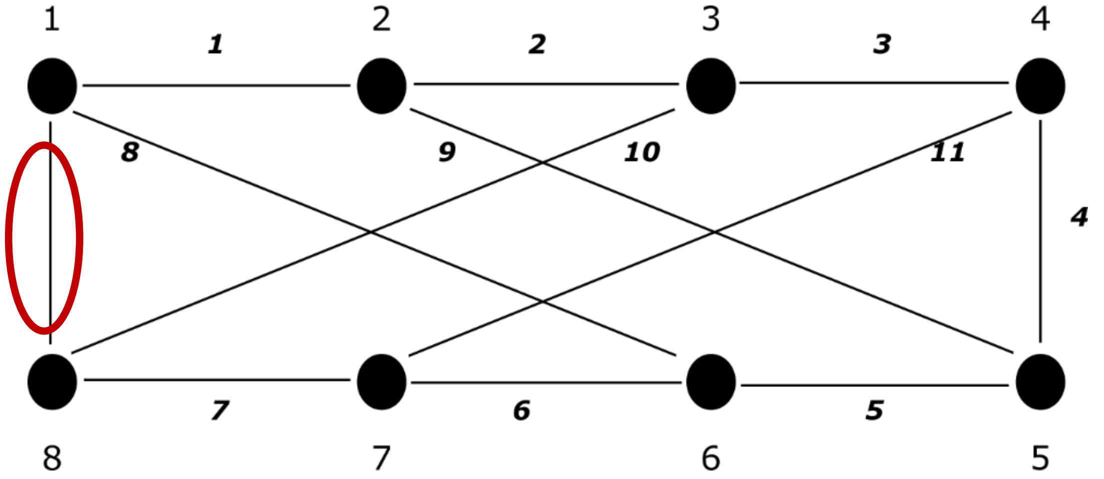
FG is parallel to DE





$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 2
\end{pmatrix}$$

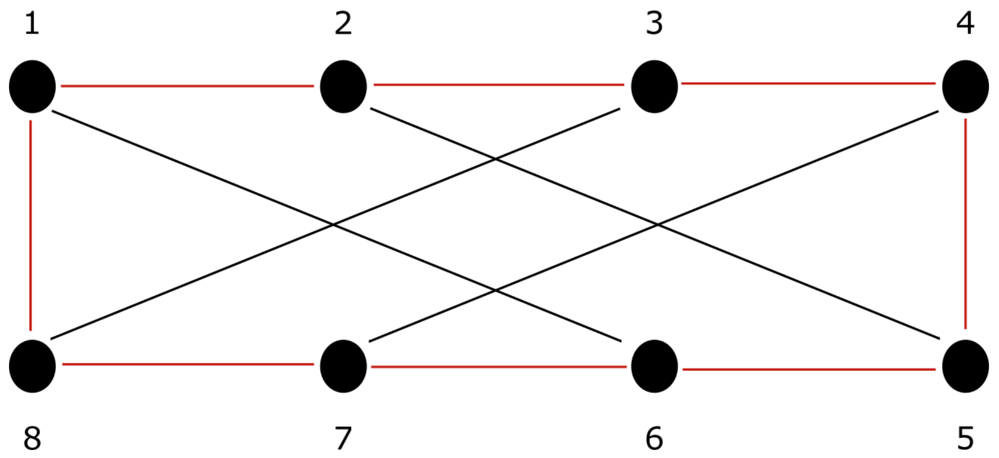
~~FG is parallel to DE~~ *FG and ED are the same direction*



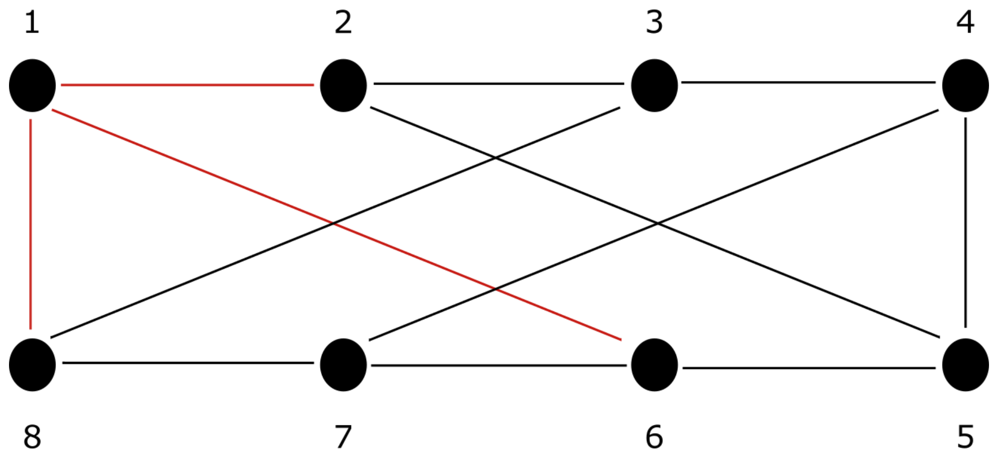
Definitions

If each column of M has exactly two non-zero entries, then the shape hypergraph is a graph
Call it the **shape graph** of M.

Graph Characteristics



Hamiltonian –a cycle contains every vertex



Cubic – all vertices degree 3

Definitions

Let e_{ij} be the directed edge of shape graph G corresponding to column k of matrix M.
Define the directed edge weight w_{ij} to be

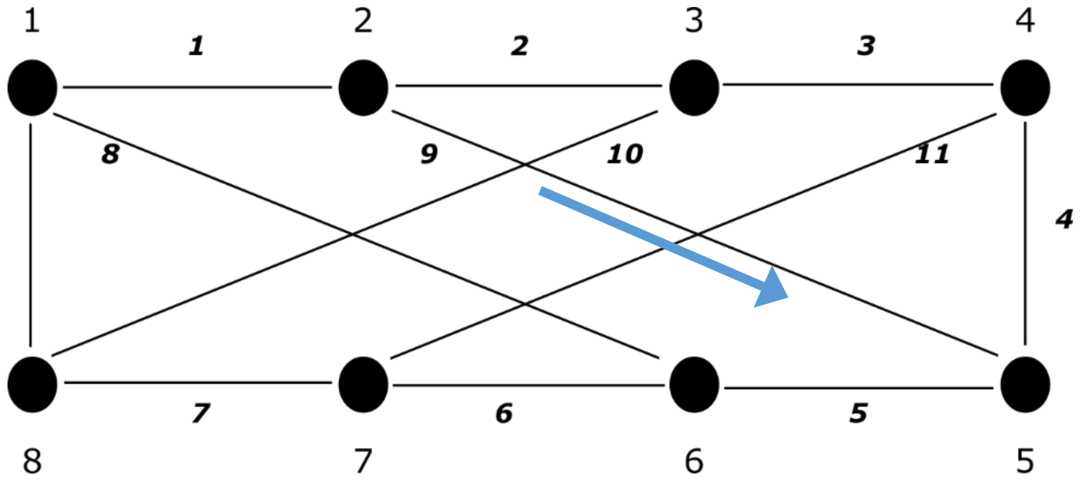
$$\frac{-M_{ik}}{M_{jk}}$$

Clearly

$$w_{ji} = \frac{1}{w_{ij}}$$

Call $\{w_{ij}\}$ the $\boldsymbol{directed\ edge\ weights}$ of M.

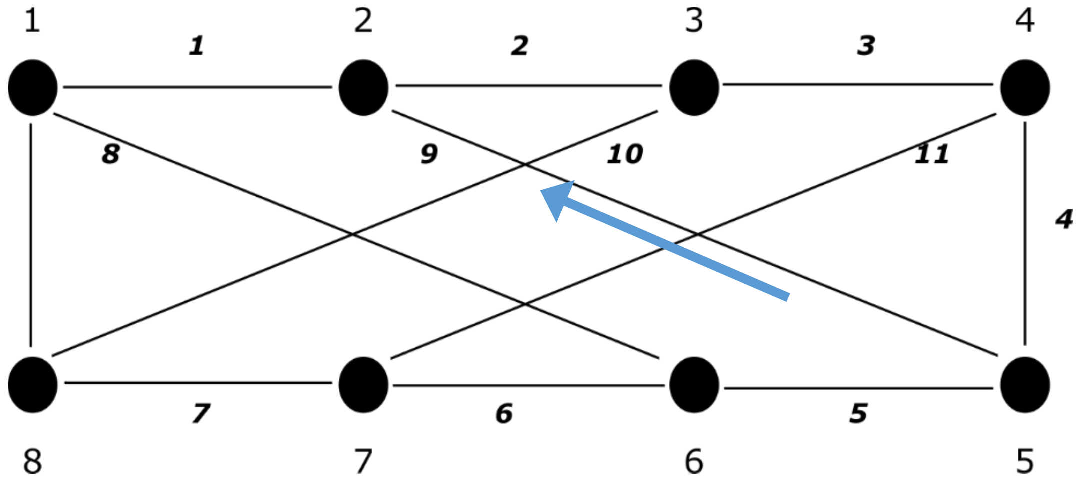
Directed Edge Weights



$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & -1 \end{pmatrix}$$

Weight: $\frac{1}{2}$

Directed Edge Weights



$$
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & -1
\end{pmatrix}
$$

Weight: 2

Theorem

Given a matrix M with Hamiltonian shape graph G the row space of M is linearly dependent if and only if the product of the directed edge weights around any cycle of G is 1.

Proof Sketch

This is an expression of the fact that the row space and the column space have the same dimension.

A Hamiltonian path gives us a linearly independent set of column vectors of size m-1.

If the rank is less than m, then all other column vectors must be expressible as a linear combination of these.

This yields the condition on cycles (via Gaussian Elimination).

Definition

Definition: A ***bisector matrix*** has exactly three non-zero elements in each row, and those elements have the values -1, -1, 2 in some order

By definition, G is cubic (all vertices have degree 3).

Corollary

Let M be a bisector matrix with Hamiltonian shape graph G.
If the row space of M is linearly dependent then G has only even cycles.

i.e. G is bicubic

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Let M be a bisector matrix with Hamiltonian shape graph G.
If the row space of M is linearly dependent then G has only even cycles.

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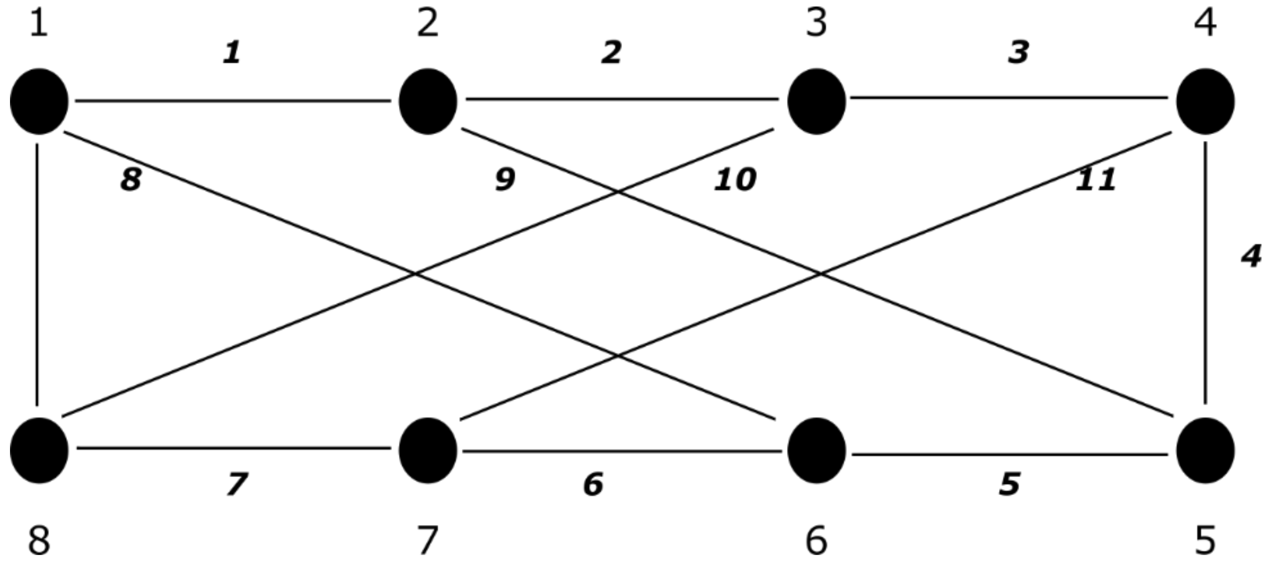
Proof: Edge weights are -1, 2 or ½

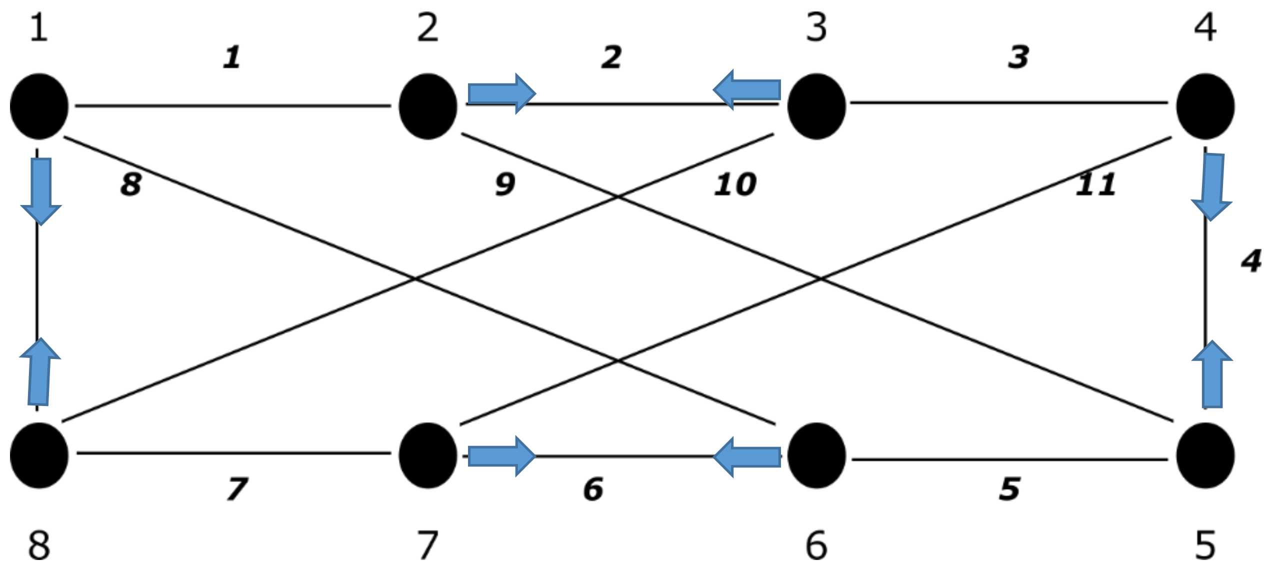
Number of Bicubic Graphs on n vertices

n	Cubic Hamiltonian	Bicubic
6	2	1
8	5	1
10	17	2
12	80	5

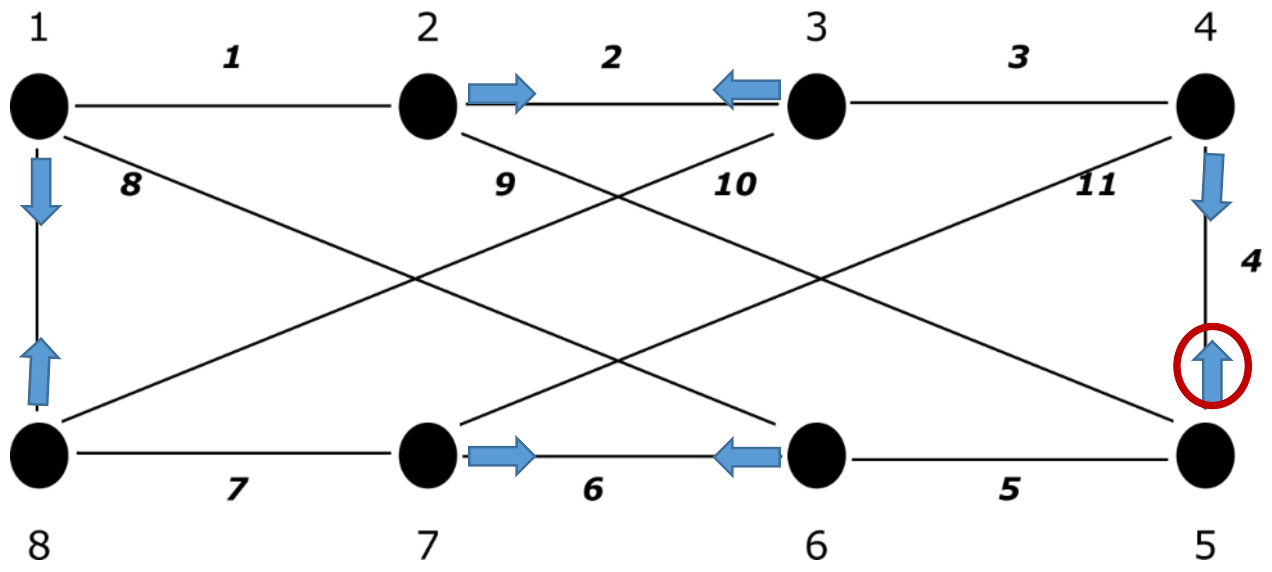
$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$



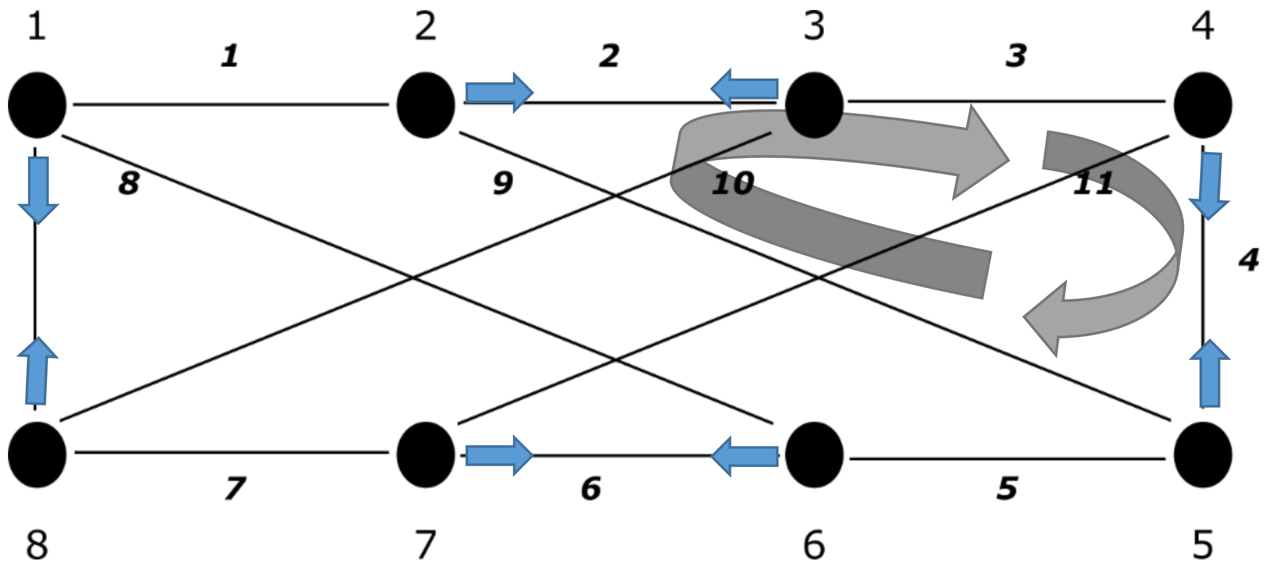
$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 2
\end{pmatrix}$$

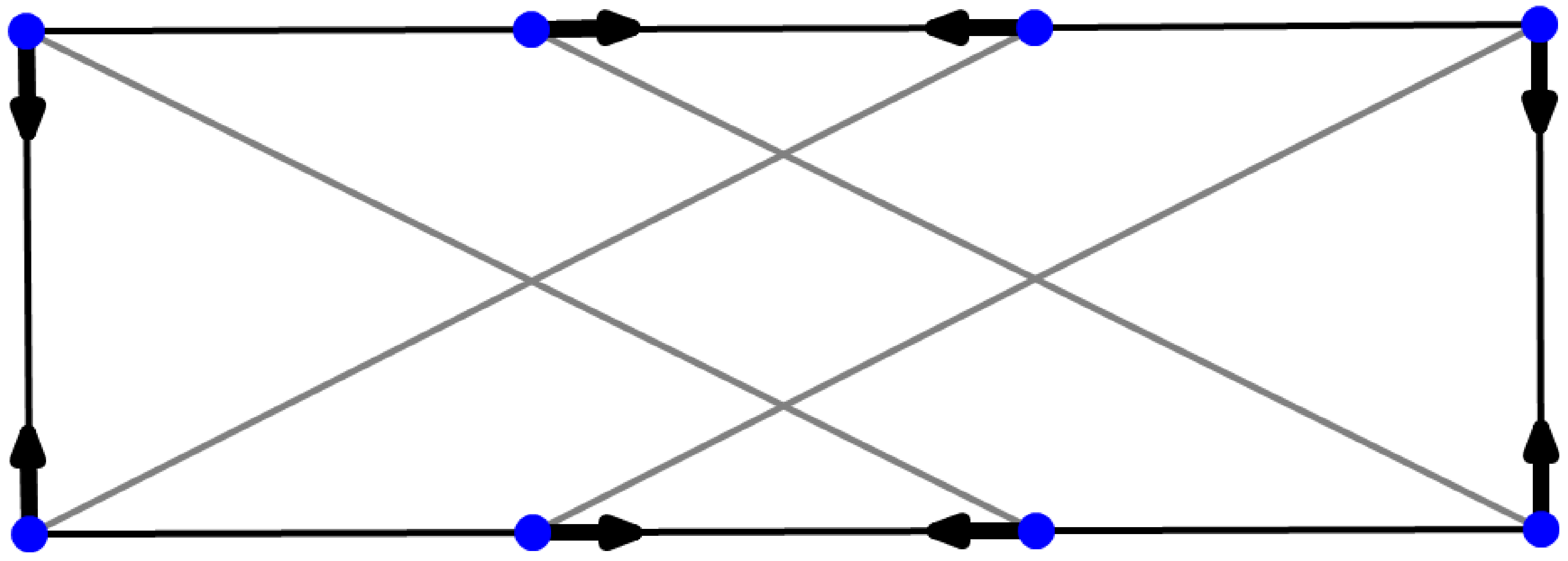


$$
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 2
\end{pmatrix}
$$



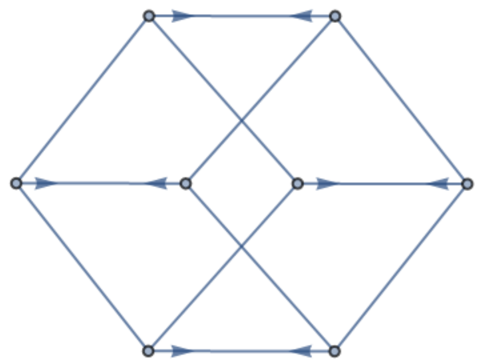
$$k_2 k_3 k_4 \frac{1}{k_9} = 1$$

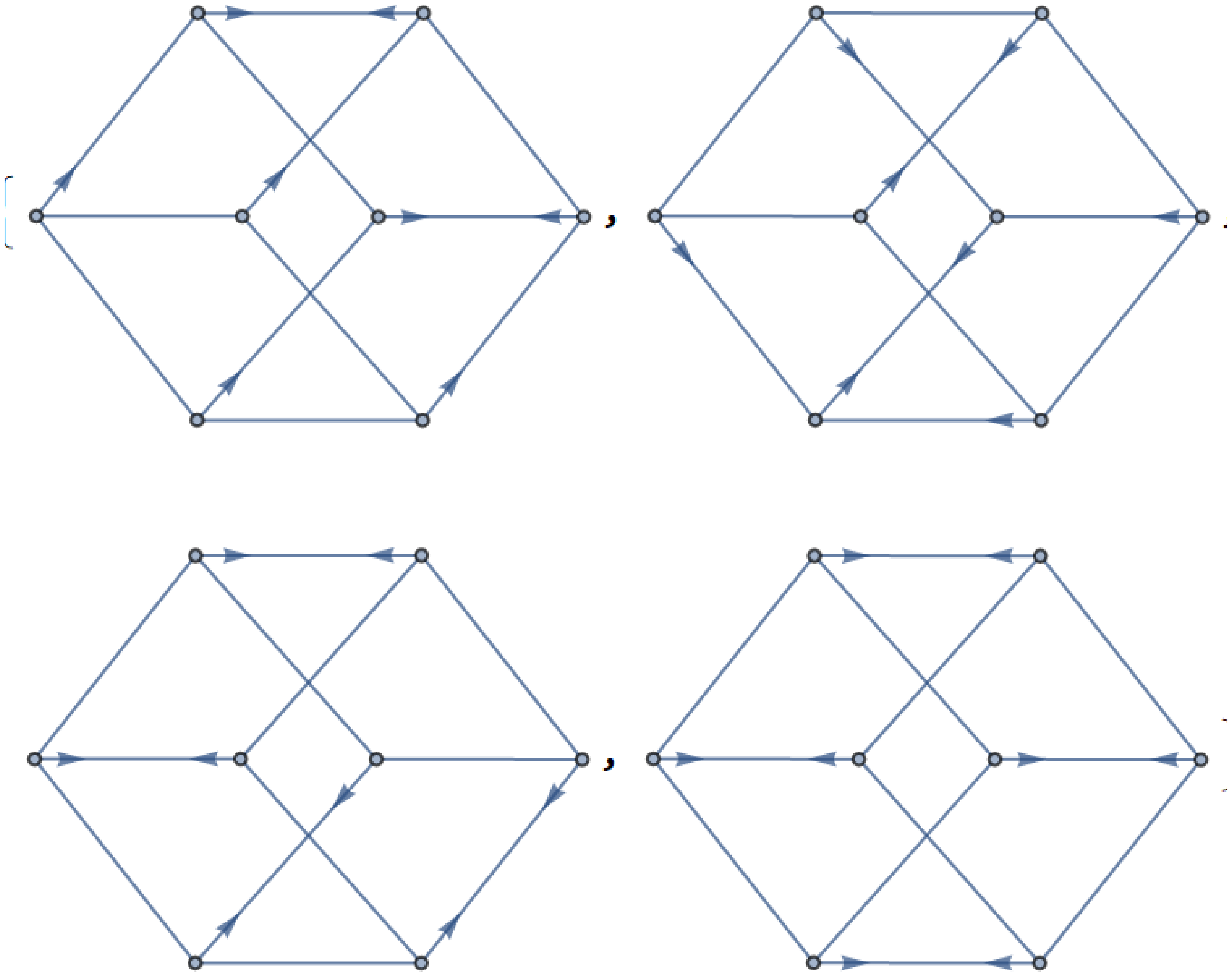


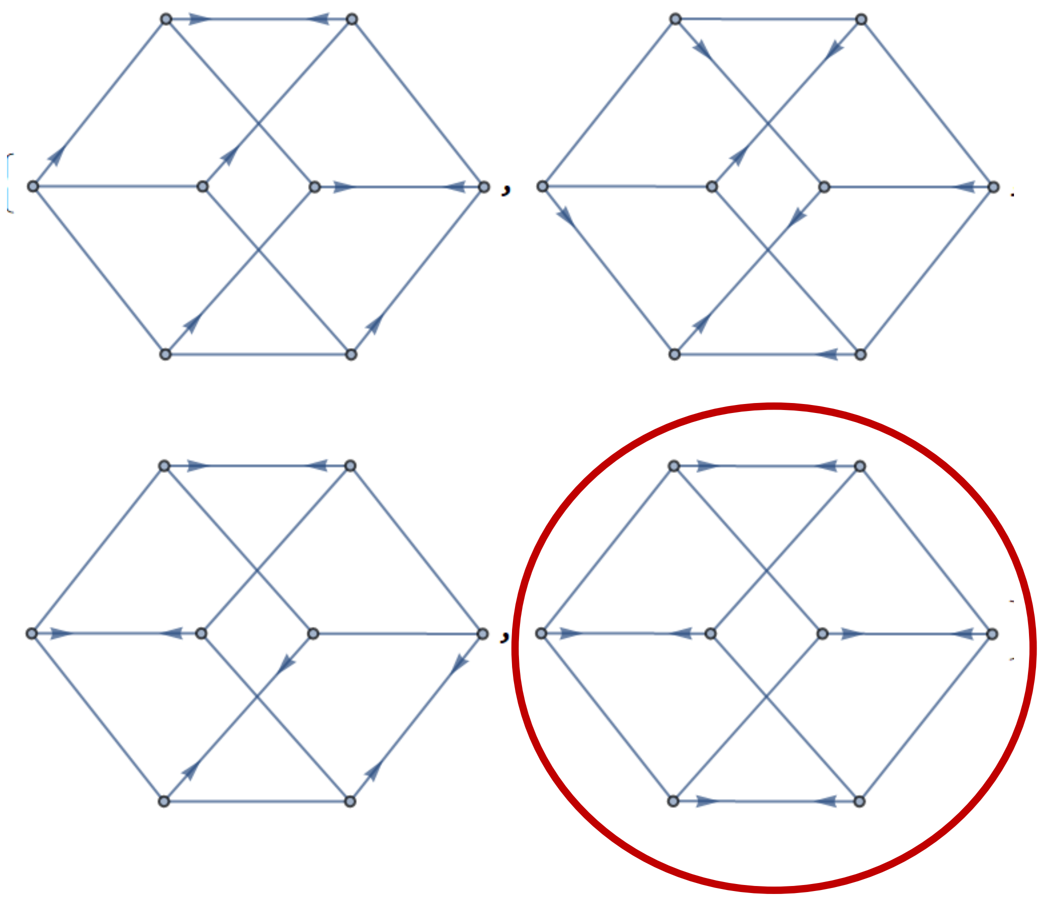


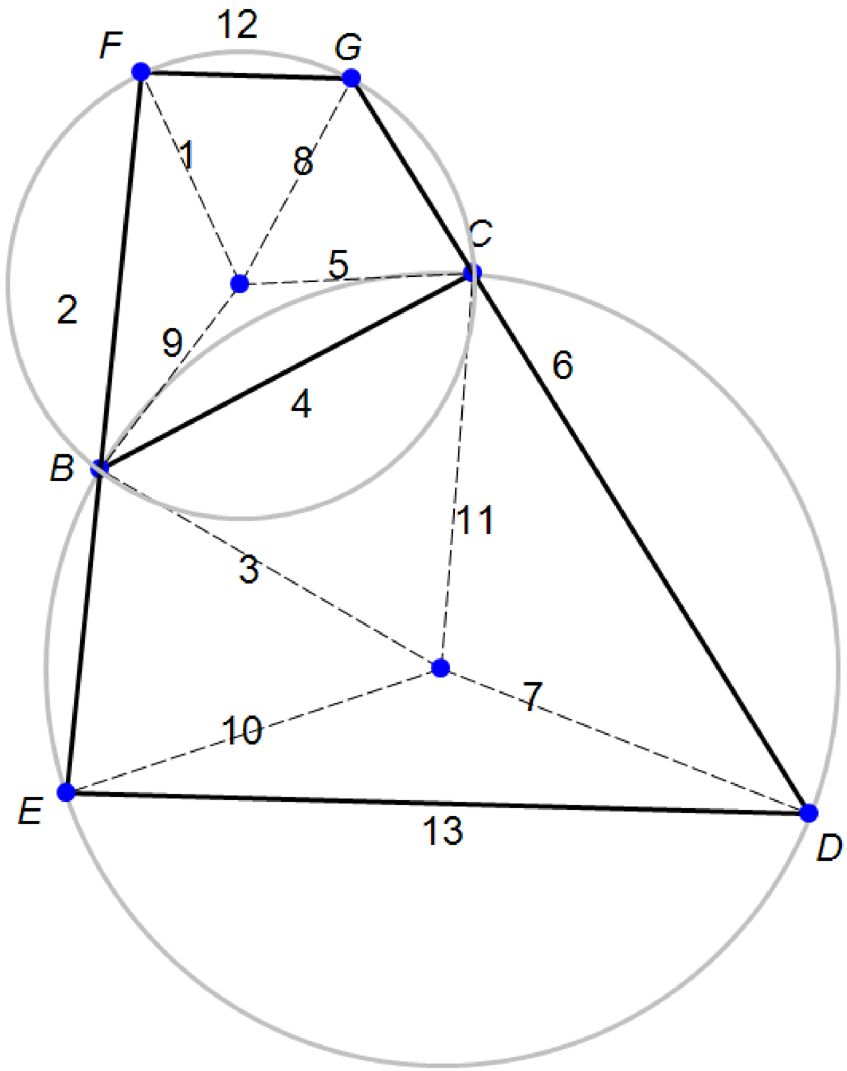
Recreational Math Problem

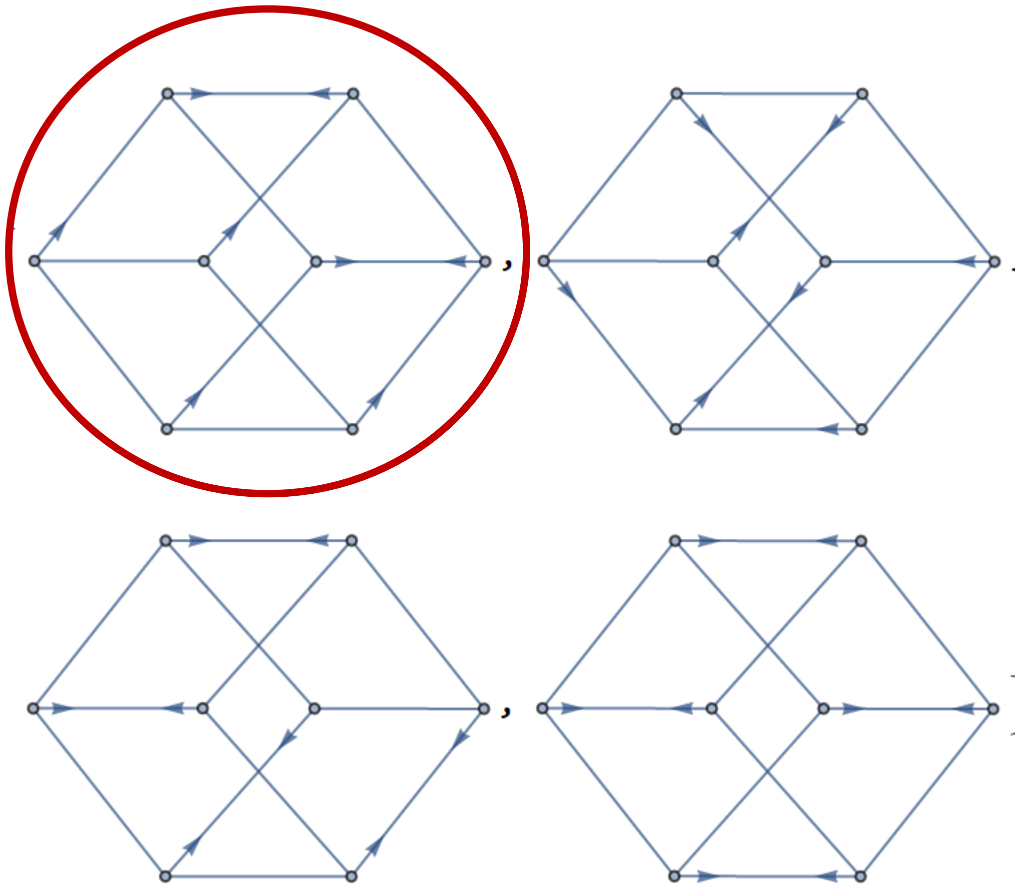
How many ways can you arrange the arrows and maintain the cycle property

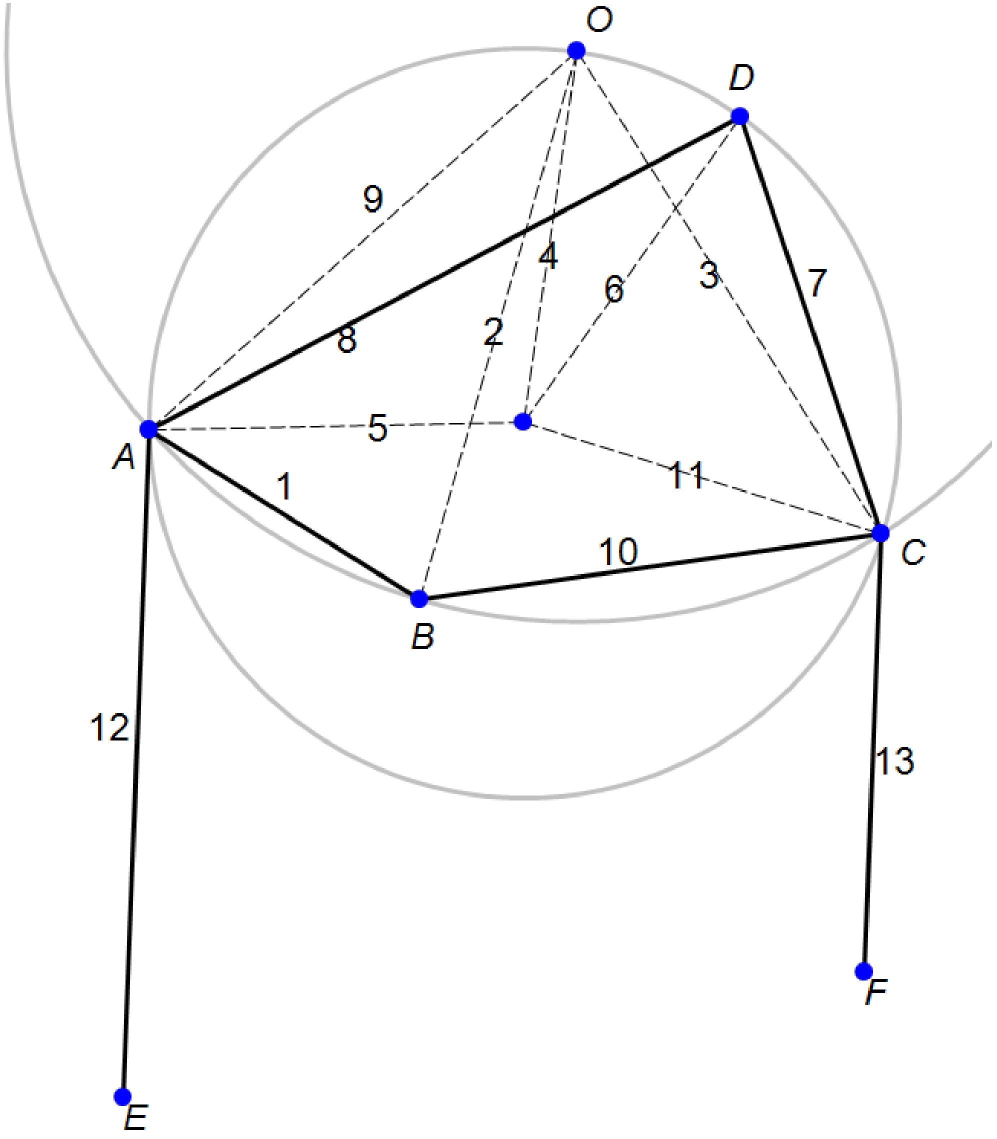







F
G
C
B
E
D
12
1
8
5
2
9
4
6
11
3
7
10
13





O
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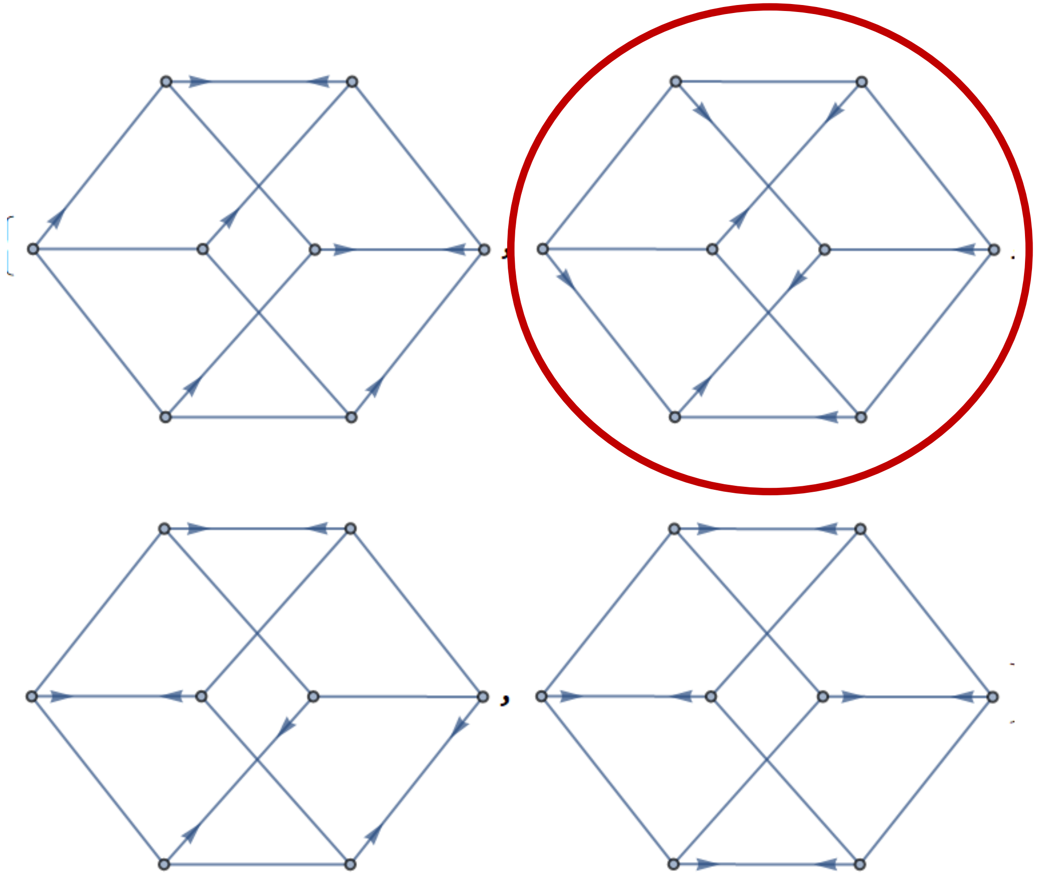

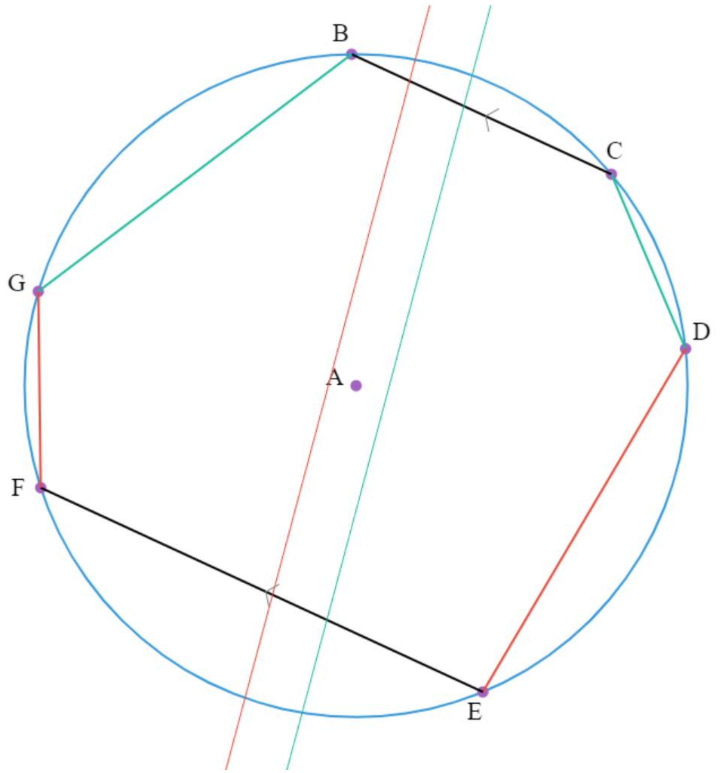

B
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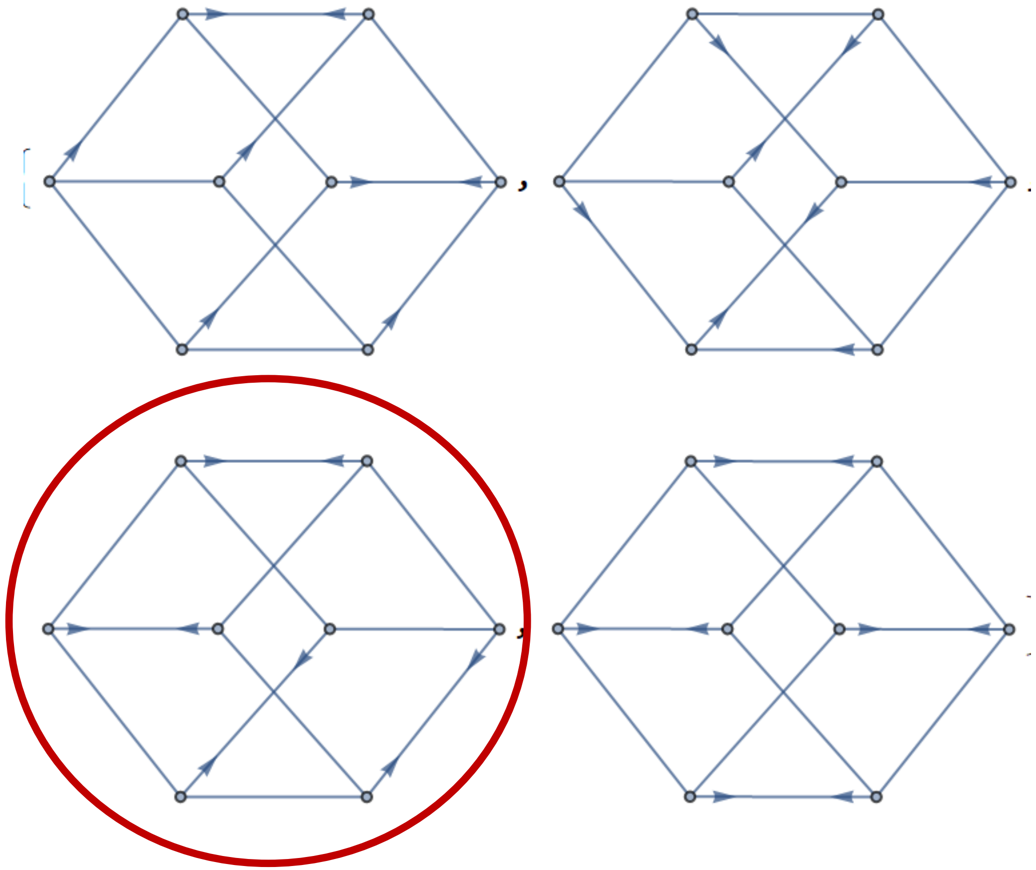

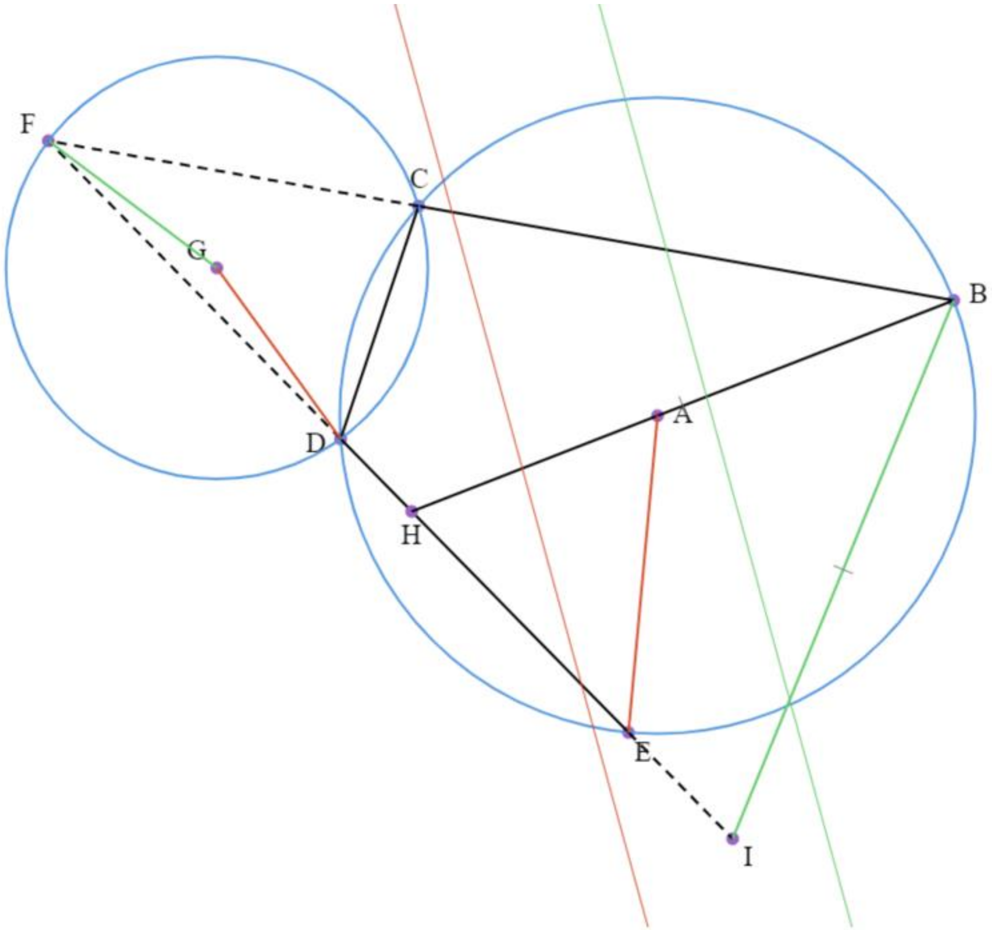
F
C
G
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D
H
E
I

Algorithm

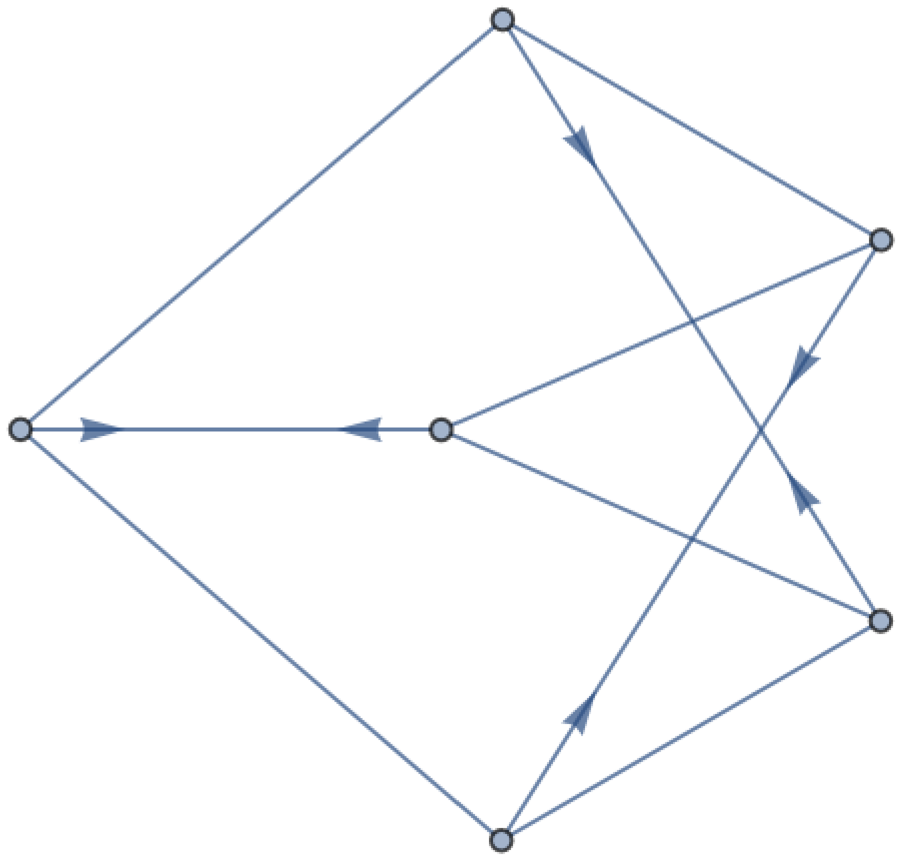
To find all non full rank bisection matrices with given bicubic shape graph.

- Start with the adjacency matrix of a bicubic graph.
- Try all possible locations for the 2 in each row.
- Check that cycles have weight product 1.

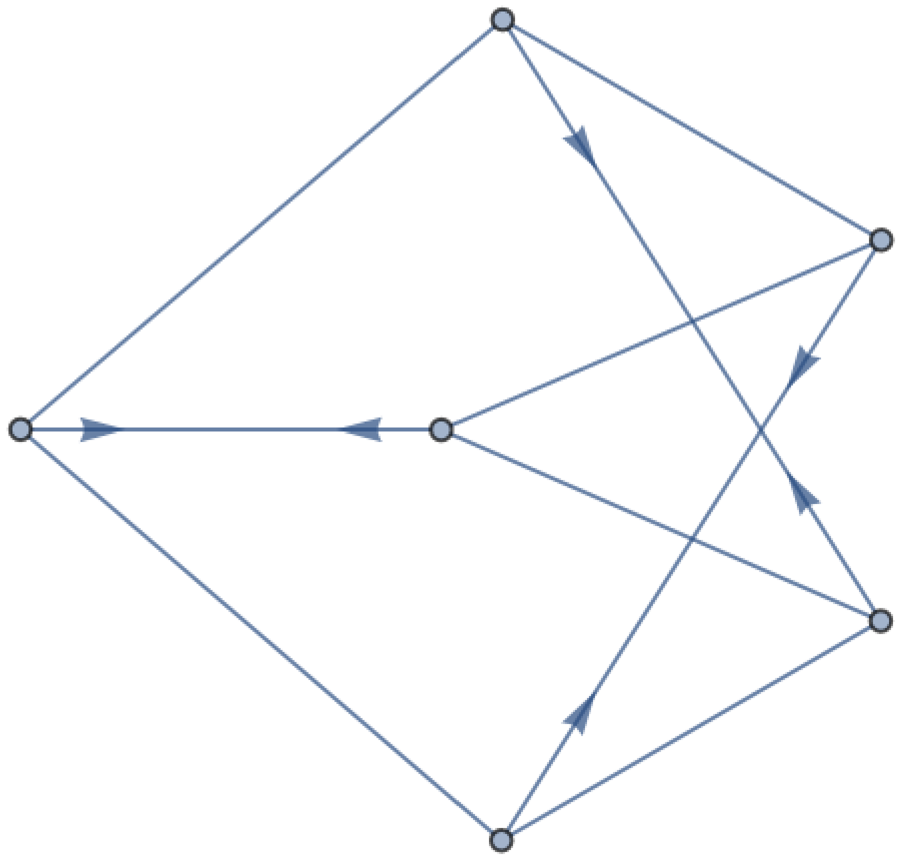
Remove symmetries using the graph automorphism group.

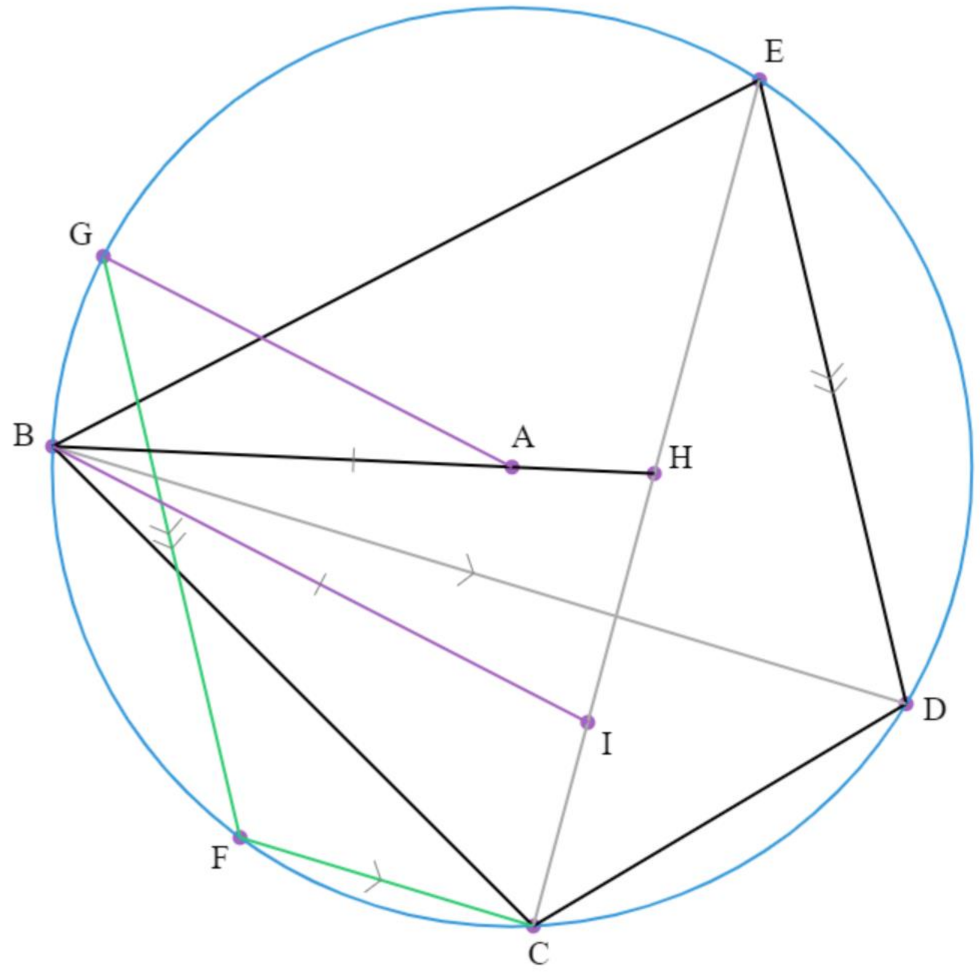
Using group theory and graph theory features of Mathematica / Maple.

6 row pattern

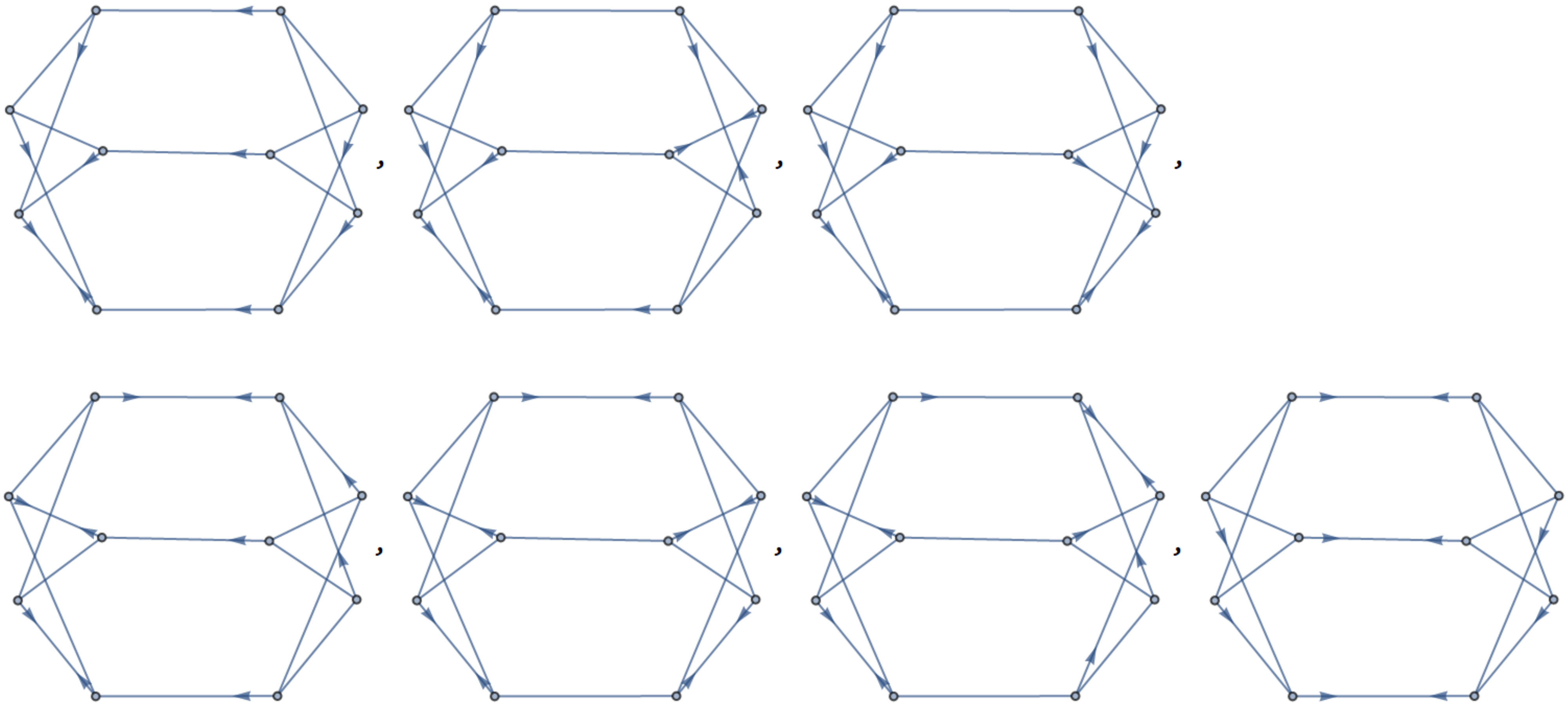


6 row pattern

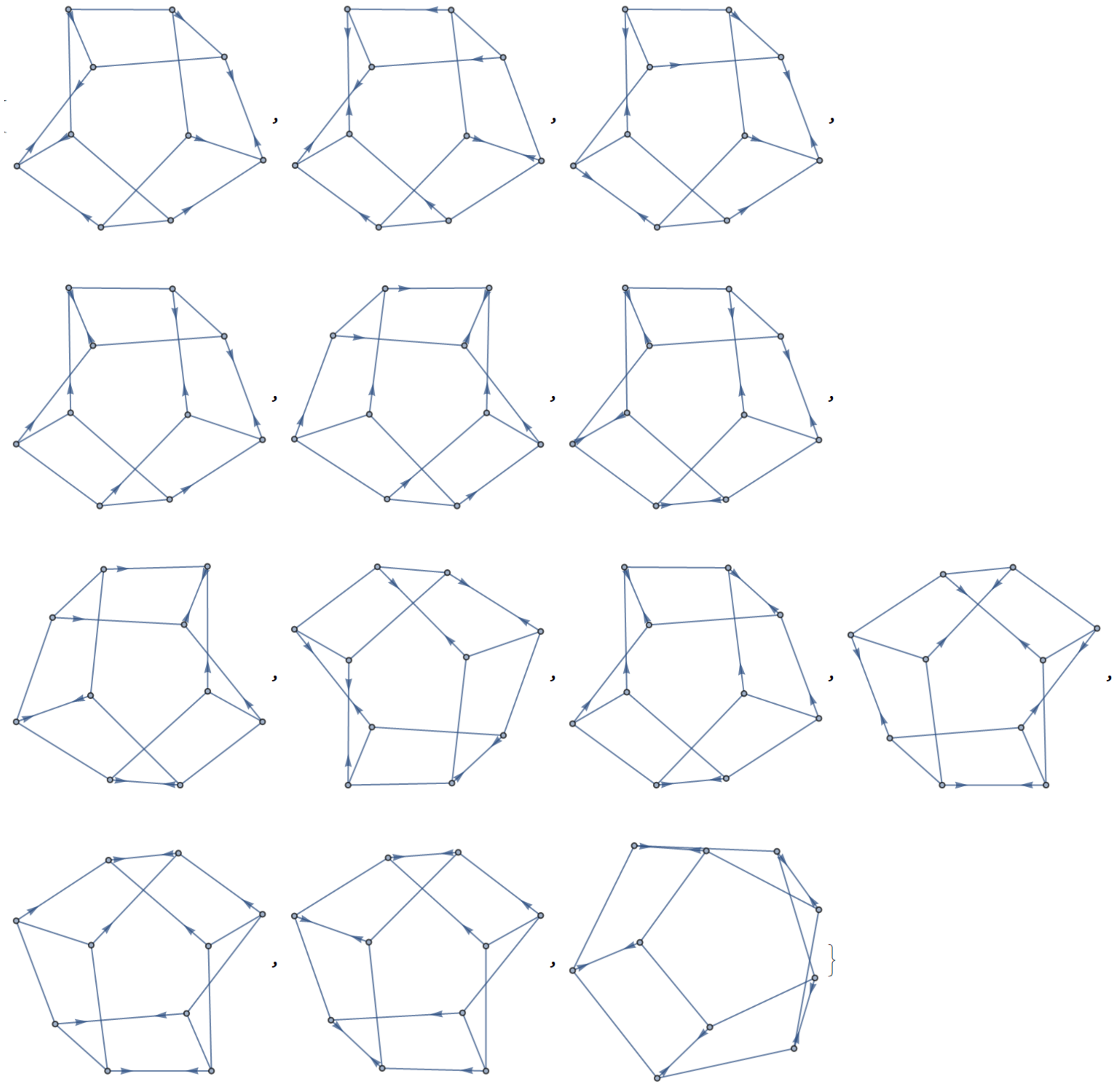




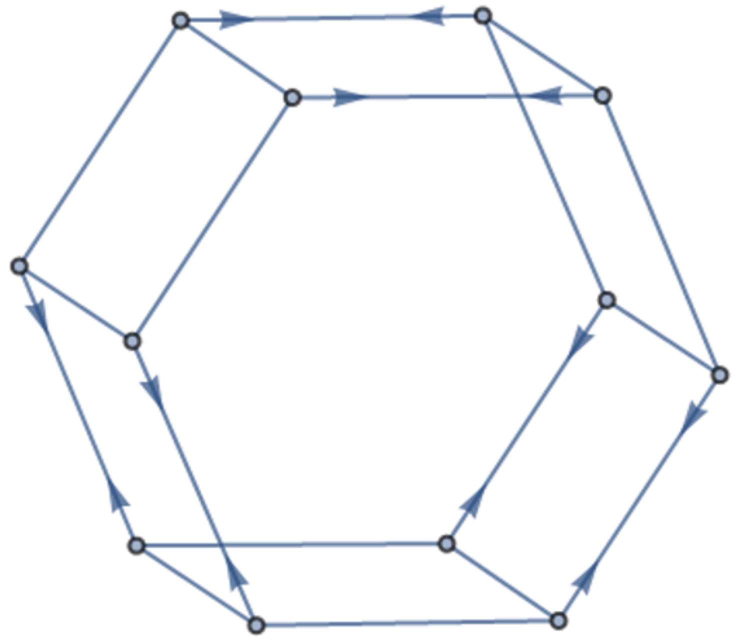
10 row patterns



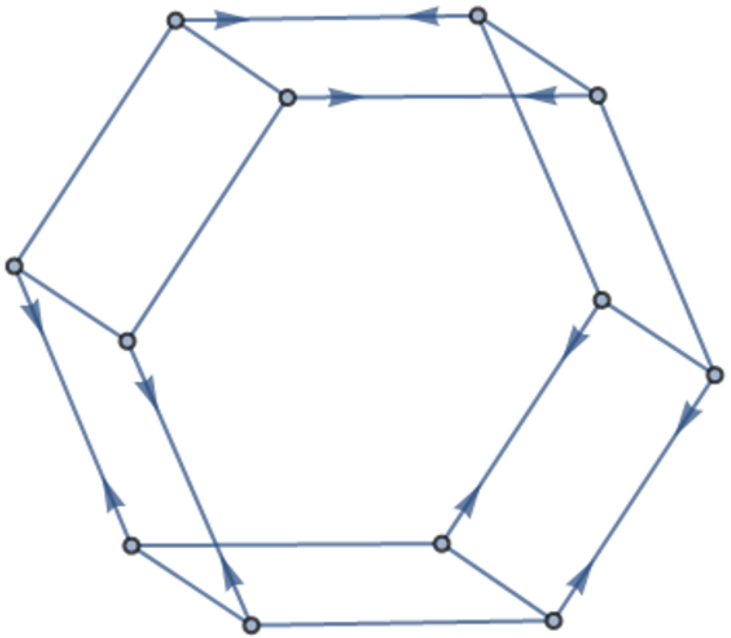
10 row patterns

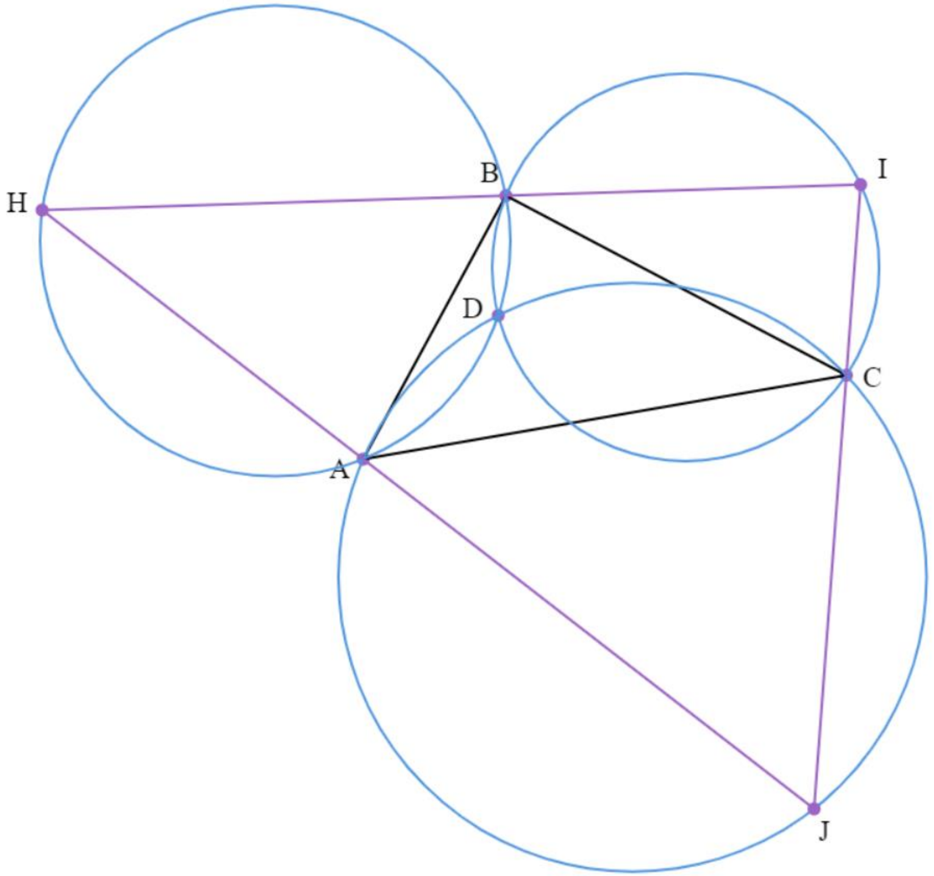


A 12 row pattern



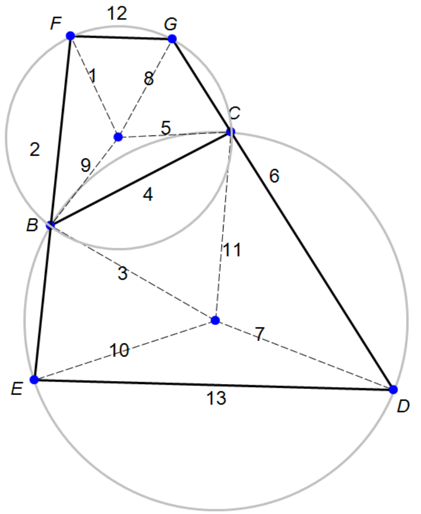
A 12 row pattern

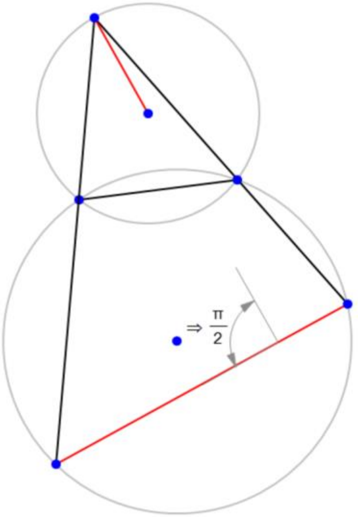


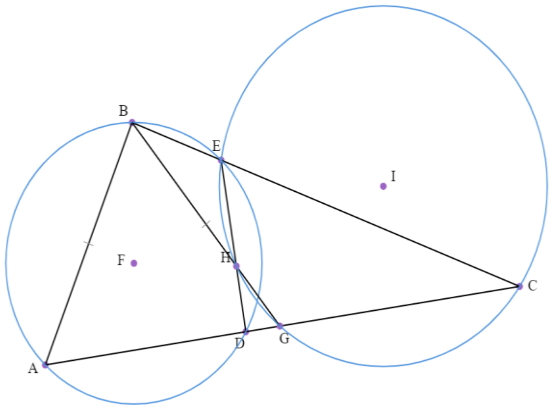


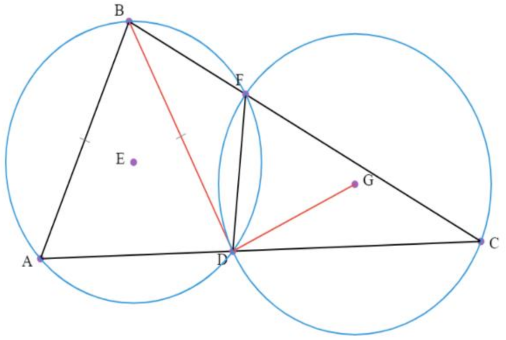
Further Work Done

There are ways to modify such matrices while maintaining their ‘theorem bearing’ property.



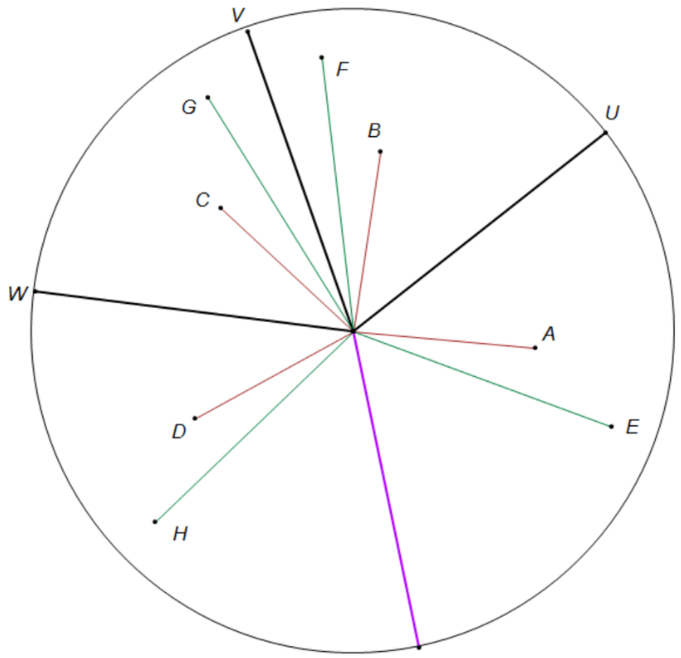






Further Work Not Yet Done

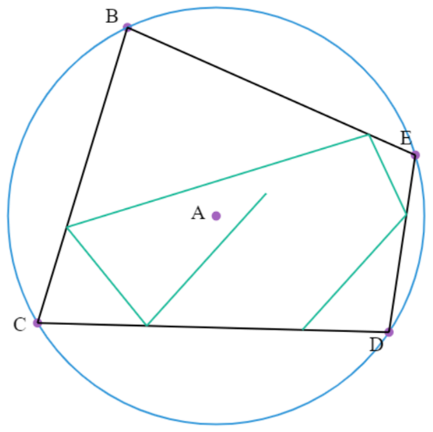
Automatically generating theorems from the matrices is easy if you are happy with theorems like this.



Let U be the angle bisector of A and B and of E and F.
Let V be the angle bisector of B and C and of F and G.
Let W be the angle bisector of C and D and of G and H.
Then A and D and E and H have the same angle bisector.

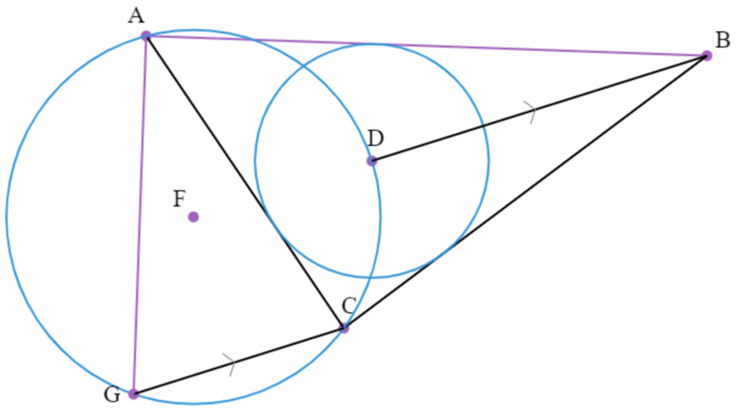
Further Work Not Yet Done

Harder (and a topic for further study) if you want theorems like this:.



Let BCDE be a cyclic quadrilateral.
A billiard ball bounces off all four walls in succession. Its final path is parallel to its initial path.

Let D be the incenter of triangle ABC.
Let G lie on the circumcircle of ADC such that CG is parallel to BD.
Angle GAB is right.



Why..

If we can generate Suduko problems of given difficulty, why not geometry proof problems?

Thank You

philt@saltire.com

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