Computing loci with GeoGebra-Discovery: some issues and suggestions

Á. Pereira-Albert, T. Recio, M. P. Vélez
(Universidad Nebrija)

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GeoGebra and GeoGebra Discovery

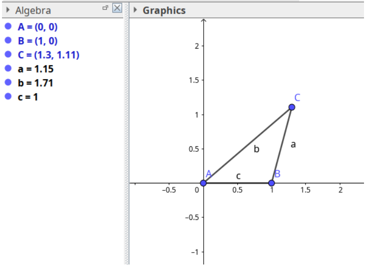
Automatic Reasoning Tools in Geogebra Discovery:

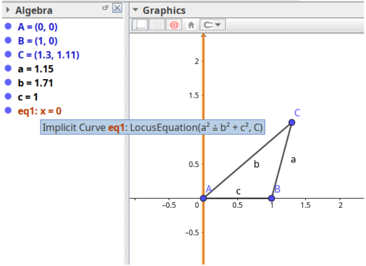
- `Prove` and `ProveDetails`.
- `Relation`.
- `Discover` and `StepwiseDiscover`.
- `ShowProof`.
- `LocusEquation`.

We are going to focus in the `LocusEquation` tool.

A first example

Given two points $A = (0,0)$ and $B = (1,0)$, find the locus of points C such that the triangle ABC satisfies Pythagoras' Theorem.





$$c_1 = 0$$

How the command works

The goal of the command

Given a geometric construction and a condition related to that construction, find the locus of points that satisfy the condition.

Algorithm

INPUT: A geometric construction (an ideal H), a condition (a polynomial P), and a point C (a set of variables appearing in the condition).
OUTPUT: An equation for C such that the condition is satisfied.
From the ideal $H + \langle P \rangle$ eliminate all variables except those belonging to the point C.

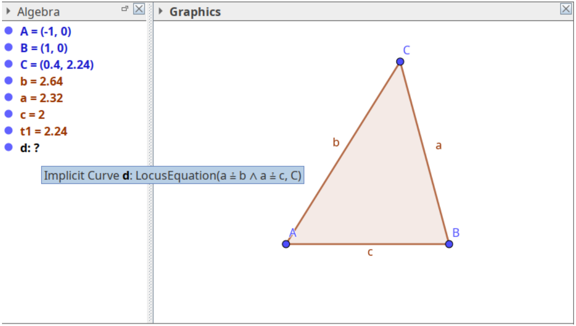
The first example

$$p_1 = a^2 - (b_1 - c_1)^2 - (b_2 - c_2)^2$$
$$p_2 = b^2 - (a_1 - c_1)^2 - (a_2 - c_2)^2$$
$$p_3 = c^2 - (a_1 - b_1)^2 - (a_2 - b_2)^2$$
$$H = \langle p_1, p_2, p_3, a_1, a_2, b_1 - 1, b_2 \rangle$$
$$P = a^2 - b^2 - c^2$$

$(H + \langle P \rangle) \cap \mathbb{C}[c_1, c_2] = \langle c_1 \rangle$

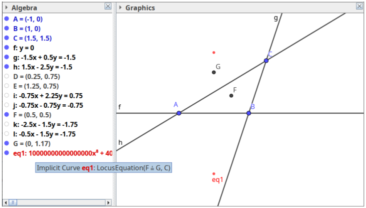
Assessing several conditions at the same time

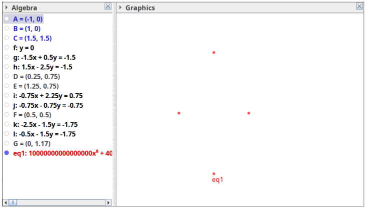
Given two points $A = (-1, 0)$ and $B = (1, 0)$, find the locus of points such that the triangle ABC is equilateral.



Non principal locus ideals and degenerate components

Given two points $A = (-1, 0)$ and $B = (1, 0)$, find the locus of points such that the triangle ABC is equilateral (the orthocenter and the barycenter are equal).





After eliminating:

$$\langle x\,y, 3x^2 + y^2 - 3, y^3 - 3y \rangle =$$
$$\langle x, y - \sqrt{3} \rangle \cap \langle x, y + \sqrt{3} \rangle \cap \langle x - 1, y \rangle \cap \langle x + 1, y \rangle$$

Non principal locus ideals and degenerate components

$$\langle x\,y, 3x^2 + y^2 - 3, y^3 - 3y\rangle =$$
$$\langle x, y - \sqrt{3}\rangle \cap \langle x, y + \sqrt{3}\rangle \cap \langle x - 1, y\rangle \cap \langle x + 1, y\rangle$$

$\langle x - 1, y\rangle$ and $\langle x + 1, y\rangle$ correspond to $A = C$ and $B = C$ respectively, which are degenerate instances of the construction.

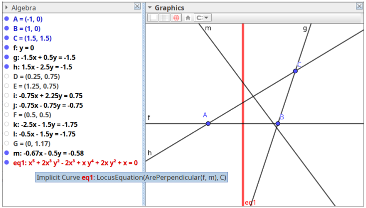
The intersection of ideals turns into the equation:

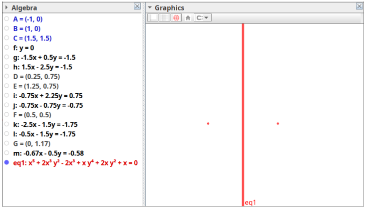
$$((x+1)^2+y^2)((x-1)^2+y^2)(x^2+(y-\tfrac{17321}{10000})^2)(x^2+(y+\tfrac{17321}{10000})^2) = 0$$

And is multiplied by 10^{16} to get rid of the denominators.

Non principal locus ideals and degenerate components

Given two points $A = (-1, 0)$ and $B = (1, 0)$, find the locus for C such that the Euler line of the triangle ABC is perpendicular to the side AB.



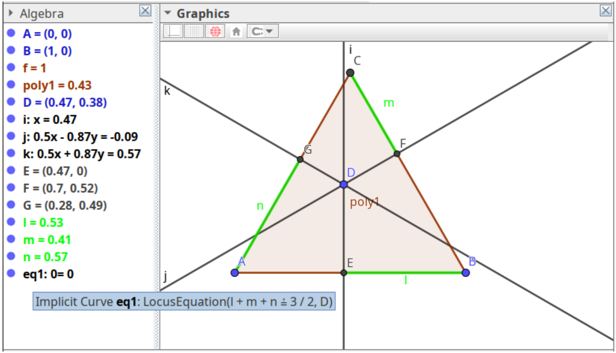


After eliminating:

$$\langle x\,y, x^3 - x \rangle = \langle x \rangle \cap \langle x - 1, y \rangle \cap \langle x + 1, y \rangle$$

Real semialgebraic locus

Given an equilateral triangle ABC and a point D, compute the projections E, F and G of D onto each of the sides of ABC, and find the locus of D such that $AG + CF + BE$ equals the semiperimeter of ABC.



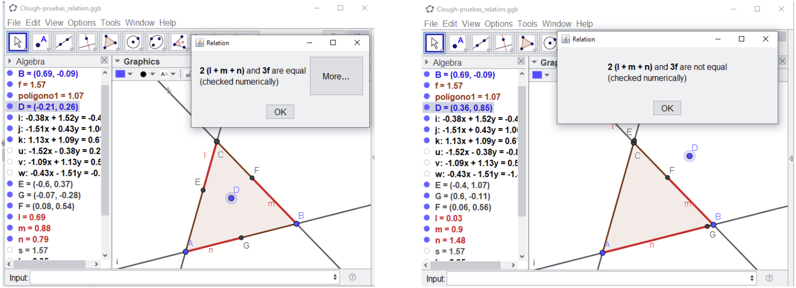
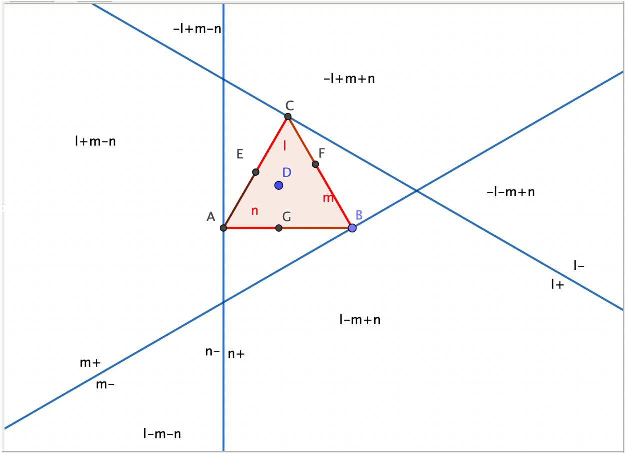


Figure: GeoGebra "numerical" answer to `Relation(2*(l+m+n),3*f)`, where f is the length of segment AB, for D inside (left) and outside (right) the triangle.

Real semialgebraic locus

The actual locus:



Discussion of the examples

What we have found in the examples:

1. The `LocusEquation` commmand doesn't admit inputting several conditions at the same time.
2. Degenerate components arise in some examples.
3. Some problems require real algebraic geometry algorithms.

Conclussions

We propose some improvements:

1. Enabling the input of several conditions at the same time.
2. Including some non-degeneracy conditions to prevent degenerate components.
3. Allowing the user to see all of the generators of the elimination ideal.
4. Showing a warning message when the computation requires real quantifier elimination.

References

M. Abánades, F. Botana, A Montes, and T. Recio, *An algebraic taxonomy for locus computation in dynamic geometry*, Computer-Aided Design **56** (2014), 22–33,DOI 10.1016/j.cad.2014.06.008.

F. Botana, and M. Abánades, *Automatic Deduction in (Dynamic) Geometry: Loci Computation*, Computational Geometry **47** (2014), no. 1, 75–89.

Recio, T., Vélez M.P., *Augmented intelligence with GeoGebra and Maple involvement*, Electronic Proceedings of the 27th Asian Technology Conference in Mathematics (2022). Editors: Wei-Chi Yang, Miroslaw Majewski, Douglas Meade, Weng Kin Ho. Published by Mathematics and Technology, LLC (http://mathandtech.org/).

Thank you for your attention