Is a regular polygon determined by its diagonals?

Gábor Gévay[1] Benedek Kovács Zoltán Kovács[2]

[1]Bolyai Institute, University of Szeged, Hungary

[2]PHDL, Linz, Austria + RISC, Hagenberg, Austria

ADG Conference, 1 August 2025
CADE Stuttgart, Germany

Abstract

We discuss some possible characterizations of regular polygons that can be directly used in algebraic provers in automated reasoning in geometry.

Why is this interesting?

- Usually, automated reasoning in geometry is interested in classic theorems connected with points, segments, lines, triangles, circles, and angles. It is rarely, however, that regular polygons are involved. In fact, there are many interesting theorems related to regular polygons that could be proven by automated reasoning as well.
- Regular polygons look simple, but an exact definition of them may be challenging.

A possible definition

. . .by using algebraic geometric means

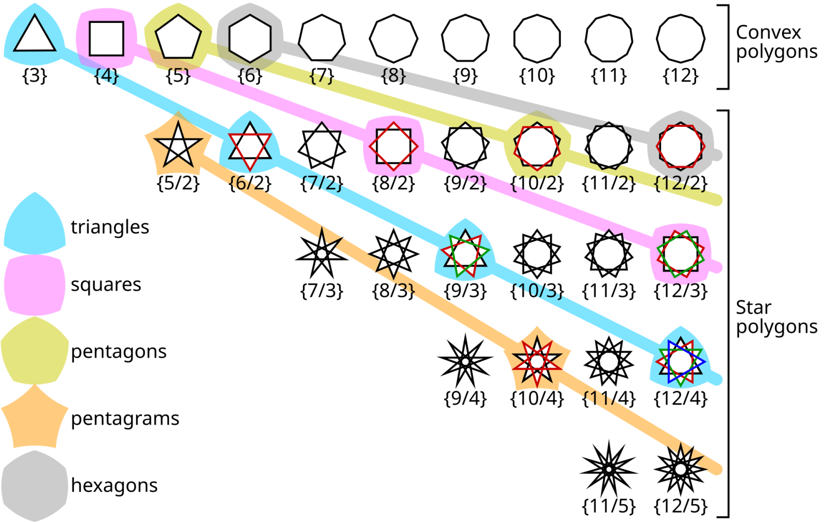
Consider the plane $\mathbb{R}^2$. Put the first two vertices of a regular n-gon in $(0,0)$ and $(1,0)$, and then set up a minimal polynomial $C_n(x) \in \mathbb{Z}[x]$ of $\cos\frac{2\pi}{n}$. Now, by setting $x^2+y^2=1$ and for the coordinates (x_i, y_i) of the regular n-gon,

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} - \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \cdot \left(\begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} - \begin{pmatrix} x_{i-2} \\ y_{i-2} \end{pmatrix} \right),$$

$i = 2, 3, \ldots, n-1$, we have a set of algebraic equations that uniquely describe a regular n-gon together with its star-regular counterparts, as a total of $\varphi(n)$ cases.

A general issue (unsolvable in algebraic geometry)

Impossible to distinguish between regular and star-regular cases



Other issues

- Multiplication of matrices may be a non-trivial concept for young learners.
- The approach also uses a non-trivial formula introduced by Watkins and Zeitlin (1993) which is based on the Chebyshev polynomials of the first kind.

Thus, it makes difficult to communicate even simple results based on regular polygons for a non-expert audience (e.g., for students or young learners).

Properties of regular polygons

...that may be used to find other characterizations

- D_1 Their sides are of equal length.
- D_2 Their shortest diagonals are of equal length.
- D_3 Their second-shortest diagonals are of equal length.
- $D_{\dots}$ ...
- $D_{\lfloor \frac{n}{2} \rfloor}$ Their longest diagonals are of equal length.
- N_i The ith point is different from the first point ($P_i \neq P_0$, $0 \leq i < n$).
- N_{ij} The ith and jth points are different ($P_i \neq P_j$, $0 \leq i, j < n$).
- I_1 Their interior angles are equal.

Properties of regular polygons

...that may be used to find other characterizations

- D_1 Their sides are of equal length.
- D_2 Their shortest diagonals are of equal length.
- D_3 Their second-shortest diagonals are of equal length.
- $D_{\dots}$ ...
- $D_{\lfloor \frac{n}{2} \rfloor}$ Their longest diagonals are of equal length.
- N_i The ith point is different from the first point ($P_i \neq P_0$, $0 \leq i < n$).
- N_{ij} The ith and jth points are different ($P_i \neq P_j$, $0 \leq i, j < n$).
- I_1 Their interior angles are equal. (The tan, sin or cos of their interior angles are equal.)

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ $(P_0 \neq P_1 \wedge |P_0P_1| = |P_1P_2| = |P_2P_0|)$

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ $(P_0 \neq P_1 \wedge |P_0 P_1| = |P_1 P_2| = |P_2 P_0|)$
- For $n = 4$: $N_1 \wedge N_2 \wedge D_1 \wedge D_2$ $(P_0 \neq P_1 \wedge P_0 \neq P_2 \wedge |P_0 P_1| = |P_1 P_2| = |P_2 P_3| = |P_3 P_0| \wedge |P_0 P_2| = |P_1 P_3|)$

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ ($P_0 \neq P_1 \wedge |P_0P_1| = |P_1P_2| = |P_2P_0|$)
- For $n = 4$: $N_1 \wedge N_2 \wedge D_1 \wedge D_2$ ($P_0 \neq P_1 \wedge P_0 \neq P_2 \wedge |P_0P_1| = |P_1P_2| = |P_2P_3| = |P_3P_0| \wedge |P_0P_2| = |P_1P_3|$)
- For $n = 5$: $N_1 \wedge D_1 \wedge D_2$

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ $(P_0 \neq P_1 \wedge |P_0P_1| = |P_1P_2| = |P_2P_0|)$
- For $n = 4$: $N_1 \wedge N_2 \wedge D_1 \wedge D_2$ $(P_0 \neq P_1 \wedge P_0 \neq P_2 \wedge |P_0P_1| = |P_1P_2| = |P_2P_3| = |P_3P_0| \wedge |P_0P_2| = |P_1P_3|)$
- For $n = 5$: $N_1 \wedge D_1 \wedge D_2$
- For $n = 6$: $N_1 \wedge N_2 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_3$

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ $(P_0 \neq P_1 \wedge |P_0P_1| = |P_1P_2| = |P_2P_0|)$
- For $n = 4$: $N_1 \wedge N_2 \wedge D_1 \wedge D_2$ $(P_0 \neq P_1 \wedge P_0 \neq P_2 \wedge |P_0P_1| = |P_1P_2| = |P_2P_3| = |P_3P_0| \wedge |P_0P_2| = |P_1P_3|)$
- For $n = 5$: $N_1 \wedge D_1 \wedge D_2$
- For $n = 6$: $N_1 \wedge N_2 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_3$
- For $n = 7$: $N_1 \wedge D_1 \wedge D_2$

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ $(P_0 \neq P_1 \wedge |P_0P_1| = |P_1P_2| = |P_2P_0|)$
- For $n = 4$: $N_1 \wedge N_2 \wedge D_1 \wedge D_2$ $(P_0 \neq P_1 \wedge P_0 \neq P_2 \wedge |P_0P_1| = |P_1P_2| = |P_2P_3| = |P_3P_0| \wedge |P_0P_2| = |P_1P_3|)$
- For $n = 5$: $N_1 \wedge D_1 \wedge D_2$
- For $n = 6$: $N_1 \wedge N_2 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_3$
- For $n = 7$: $N_1 \wedge D_1 \wedge D_2$

Is there a general rule?

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ $(P_0 \neq P_1 \wedge |P_0P_1| = |P_1P_2| = |P_2P_0|)$
- For $n = 4$: $N_1 \wedge N_2 \wedge D_1 \wedge D_2$ $(P_0 \neq P_1 \wedge P_0 \neq P_2 \wedge |P_0P_1| = |P_1P_2| = |P_2P_3| = |P_3P_0| \wedge |P_0P_2| = |P_1P_3|)$
- For $n = 5$: $N_1 \wedge D_1 \wedge D_2$
- For $n = 6$: $N_1 \wedge N_2 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_3$
- For $n = 7$: $N_1 \wedge D_1 \wedge D_2$

Is there a general rule? Can we find simple characterizations?

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ $(P_0 \neq P_1 \wedge |P_0P_1| = |P_1P_2| = |P_2P_0|)$
- For $n = 4$: $N_1 \wedge N_2 \wedge D_1 \wedge D_2$ $(P_0 \neq P_1 \wedge P_0 \neq P_2 \wedge |P_0P_1| = |P_1P_2| = |P_2P_3| = |P_3P_0| \wedge |P_0P_2| = |P_1P_3|)$
- For $n = 5$: $N_1 \wedge D_1 \wedge D_2$
- For $n = 6$: $N_1 \wedge N_2 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_3$
- For $n = 7$: $N_1 \wedge D_1 \wedge D_2$

Is there a general rule? Can we find simple characterizations? How about restricting non-degeneracy conditions to N_d where $d \mid n$?

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ $(P_0 \neq P_1 \wedge |P_0P_1| = |P_1P_2| = |P_2P_0|)$
- For $n = 4$: $N_1 \wedge N_2 \wedge D_1 \wedge D_2$ $(P_0 \neq P_1 \wedge P_0 \neq P_2 \wedge |P_0P_1| = |P_1P_2| = |P_2P_3| = |P_3P_0| \wedge |P_0P_2| = |P_1P_3|)$
- For $n = 5$: $N_1 \wedge D_1 \wedge D_2$
- For $n = 6$: $N_1 \wedge N_2 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_3$
- For $n = 7$: $N_1 \wedge D_1 \wedge D_2$

Is there a general rule? Can we find simple characterizations? How about restricting non-degeneracy conditions to N_d where $d \mid n$? How to prove if a generalization is correct "for all n"?

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ $(P_0 \neq P_1 \wedge |P_0P_1| = |P_1P_2| = |P_2P_0|)$
- For $n = 4$: $N_1 \wedge N_2 \wedge D_1 \wedge D_2$ $(P_0 \neq P_1 \wedge P_0 \neq P_2 \wedge |P_0P_1| = |P_1P_2| = |P_2P_3| = |P_3P_0| \wedge |P_0P_2| = |P_1P_3|)$
- For $n = 5$: $N_1 \wedge D_1 \wedge D_2$
- For $n = 6$: $N_1 \wedge N_2 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_3$
- For $n = 7$: $N_1 \wedge D_1 \wedge D_2$

Is there a general rule? Can we find simple characterizations? How about restricting non-degeneracy conditions to N_d where $d \mid n$? How to prove if a generalization is correct "for all n"? (For a given n, we can always use computer algebra and utilize elimination from algebraic geometry. This is also useful to find counterexamples for ambiguous sets of assumptions.)

Example for $n = 5$ in the CAS *Giac*

Fix P_0, P_1, use Rabinowitsch's trick

```
n:=5
// Define points:
X:=[0,1,x2,x3,x4] // we always fix P0 to (0,0) and P1 to (1,0)
Y:=[0,0,y2,y3,y4]
// List of polynomials:
p:=[undef]
sqdist(i,j):=(X[j]-X[i])^2+(Y[j]-Y[i])^2
l:=0
// Assume D1, D2, ...
for(j=1;j<n;j++) {for(k=1;k<=n/2;k++,l++)
↪  {p[l]:=sqdist(j,irem(j+k,n))-sqdist(0,k)}}
p[l]:=t
// Assume N1, ...
for(j=1;j<2;j++) {if(irem(n,j)==0) p[l]:=p[l]*sqdist(0,j);}
p[l]:=p[l]-1
el:=eliminate(p,[x3,y3,x4,y4,t])
s:=solve(el,[x2,y2])
lens:=len(s)
lenphi:=Phi(n)
lens == lenphi
```

Example for $n = 5$ in the CAS *Giac* (with partial output)

Fix P_0, P_1, use Rabinowitsch's trick

```
0>> n:=5
1>> // Define points:
2>> X:=[0,1,x2,x3,x4]
3>> Y:=[0,0,y2,y3,y4]
4>> // List of polynomials:
5>> p:=[undef]
6>> sqdist(i,j):=(X[j]-X[i])^2+(Y[j]-Y[i])^2
7>> l:=0
8>> // Assume D1, D2, ...
9>> for(j=1;j<n;j++) {for(k=1;k<=n/2;k++,l++) {p[l]:=sqdist(j,irem(j+k,n))-sqdist(0,k)}}
[(x2-1)^2+y2^2-1,(x3-1)^2+y3^2-x2^2-y2^2,(x3-x2)^2+(y3-y2)^2-1,(x4-x2)^2+(y4-y2)^2-x2^2-y2^2,
↪  (x4-x3)^2+(y4-y3)^2-1,(-x3)^2+(-y3)^2-x2^2-y2^2,(-x4)^2+(-y4)^2-1,(1-x4)^2+(-y4)^2-x2^2-y2^2]
10>> p[l]:=t
11>> // Assume N1, ...
12>> for(j=1;j<2;j++) {if(irem(n,j)==0) p[l]:=p[l]*sqdist(0,j);}
13>> p[l]:=p[l]-1
[(x2-1)^2+y2^2-1,(x3-1)^2+y3^2-x2^2-y2^2,(x3-x2)^2+(y3-y2)^2-1,(x4-x2)^2+(y4-y2)^2-x2^2-y2^2,
↪  (x4-x3)^2+(y4-y3)^2-1,(-x3)^2+(-y3)^2-x2^2-y2^2,(-x4)^2+(-y4)^2-1,(1-x4)^2+(-y4)^2-x2^2-y2^2,t-1]
14>> el:=eliminate(p,[x3,y3,x4,y4,t])
[4*x2^2-6*x2+1,4*y2^2-2*x2-1]
15>> s:=solve(el,[x2,y2])
list[[-(-4*(sqrt(1/8*(sqrt(5)+5)))^2+1)/2,sqrt(1/8*(sqrt(5)+5))],[-(-4*(-sqrt(1/8*
↪  (sqrt(5)+5)))^2+1)/2,-sqrt(1/8*(sqrt(5)+5))],[-(-4*(sqrt(1/8*(-sqrt(5)+5)))^2+1)/2,sqrt(1/8*
↪  (-sqrt(5)+5))],[-(-4*(-sqrt(1/8*(-sqrt(5)+5)))^2+1)/2,-sqrt(1/8*(-sqrt(5)+5))]]
16>> lens:=len(s)
4
17>> lenphi:=Phi(n)
4
18>> lens == lenphi
true
```

Main results

1. If $n \geq 6$, the assumption set
   - N_k for each $1 \leq k < n$,
   - D_1,
   - D_2,
   - D_3

   gives an unambiguous characterization.
2. For each n ($n \geq 3$), the assumption set
   - N_k for each $1 \leq k < n$,
   - D_1,
   - D_2,
   - $\ldots$,
   - $D_{\lfloor \frac{n}{2} \rfloor}$

   gives an unambiguous characterization.

Trivial counterexamples

- For $n = 2k$ ($k \geq 2$), the assumption $D_1 \wedge D_2 \wedge \ldots \wedge D_k$ is ambiguous, because 2 copies of a regular k-gon in a loop is also a realization.
- The same problem occurs for $n = 3k$, $n = 4k$, ...

Adding N_1, N_2, ..., N_{n-1} seems to solve this issue, but using all of these extra assumptions may be unnecessary (and an overkill).

Main results

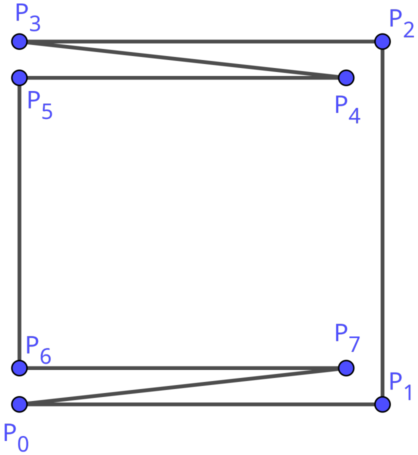
Counterexamples

- For $n = 8$, the assumption $N_1 \wedge N_2 \wedge N_4 \wedge D_1 \wedge D_3 \wedge D_4$ is ambiguous (insufficient).
- For $n = 9$, the assumption $N_1 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_4$ is insufficient.
- For $n = 10$, the assumptions
  - $N_1 \wedge N_2 \wedge N_5 \wedge D_1 \wedge D_3 \wedge D_5$,
  - $N_1 \wedge N_2 \wedge N_5 \wedge D_1 \wedge D_4 \wedge D_5$,
  - $N_1 \wedge N_2 \wedge N_5 \wedge D_2 \wedge D_4 \wedge D_5$

  are insufficient.
- For $n = 11$, the assumption $N_1 \wedge D_2 \wedge D_3 \wedge D_5$ is insufficient.

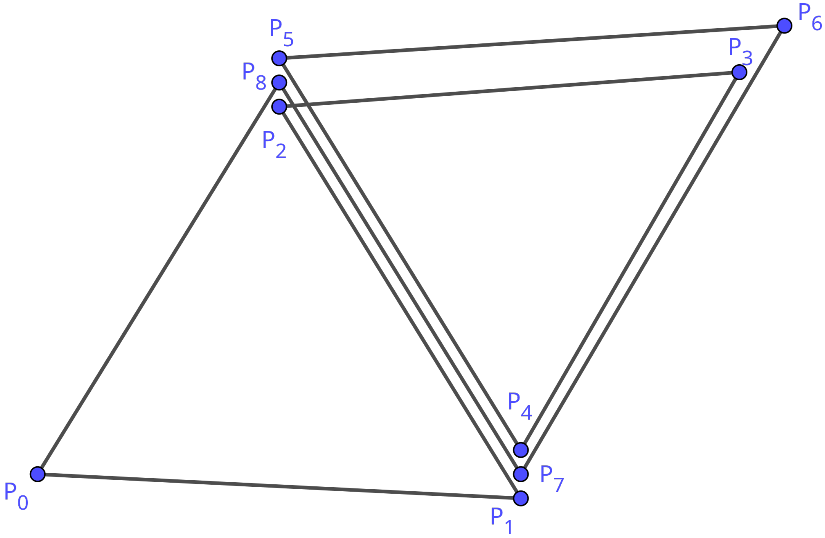
Non-trivial counterexamples

$n = 8$, $N_1 \wedge N_2 \wedge N_4 \wedge D_1 \wedge D_3 \wedge D_4$ (square)



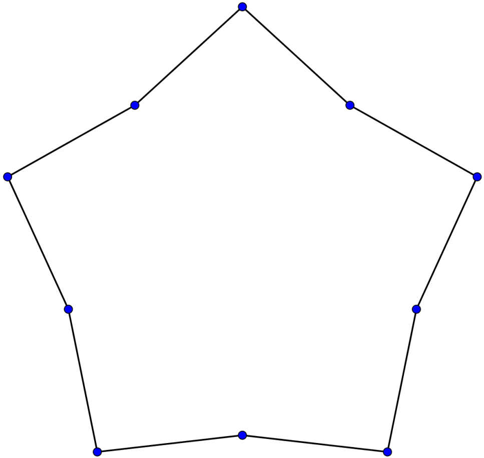
Non-trivial counterexamples

$n = 9$, $N_1 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_4$ (two regular triangles)



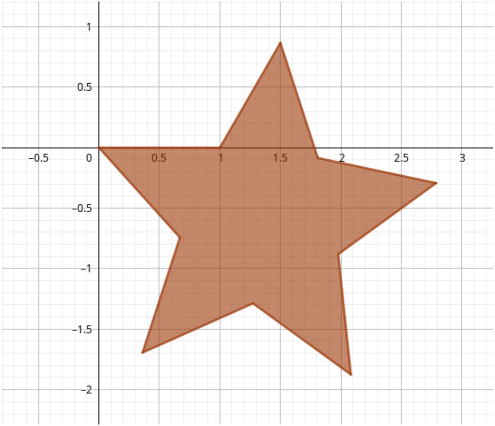
Non-trivial counterexamples

$n = 10$, $N_1 \wedge N_2 \wedge N_5 \wedge D_1 \wedge D_3 \wedge D_5$ (5-pointed stars, infinitely many variations)



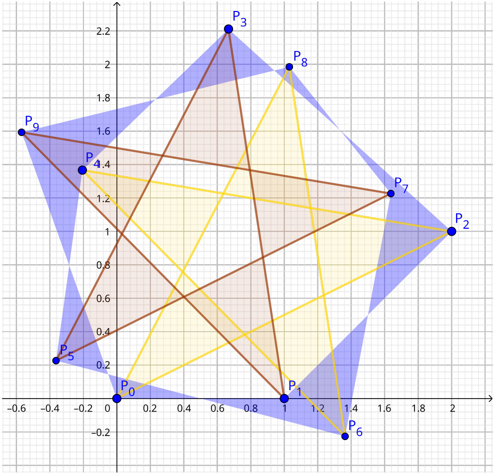
Non-trivial counterexamples

$n = 10$, $N_1 \wedge N_2 \wedge N_5 \wedge D_1 \wedge D_4 \wedge D_5$ (a 5-pointed star with 240° and 48° internal $\angle$)



Non-trivial counterexamples

$n = 10$, $N_1 \wedge N_2 \wedge N_5 \wedge D_2 \wedge D_4 \wedge D_5$ ($\cup$ of two shifted regular 5-grams, ∞ variations)



Possible issues with complex coordinates

Without non-degeneracy conditions, for $n = 11$, the assumption $D_2 \wedge D_3 \wedge D_4$ delivers possible solutions with complex coordinates, namely, for example, $P_2 = P_6 = P_7 = (0, 0)$, $P_3 = P_4 = P_5 = P_8 = P_9 = P_{10} = (1/2, i/2)$. In fact, the problem here is that the elimination theory, based on algebraic geometry, deals with coordinates from the algebraic closure of the real numbers (which is the field of the complex numbers). When being non-minimalist and setting $P_0 \neq P_i$ for each $1 < i < n$ again, this problem does not appear anymore and we obtain the regular (star) 11-gon only.

Theoretically, it would be safer to define non-overlapping of points by setting one of their coordinates different. We conjecture, however, that this safety is not required in general.

Main result (repeated)

1. If $n \geq 6$, the assumption set
   - N_k for each $1 \leq k < n$,
   - D_1,
   - D_2,
   - D_3

   gives an unambiguous characterization.
2. If $n \geq 3$, the assumption set
   - N_k for each $1 \leq k < n$,
   - D_1,
   - D_2,
   - $\ldots$,
   - $D_{\lfloor \frac{n}{2} \rfloor}$

   gives an unambiguous characterization (a weaker form).

Prop. 1 Given the points $A_0 = (0,0)$, $A_1 = (1,0)$, ..., $A_4 = (x_4, y_4)$ in the plane, $|A_0A_1| = |A_1A_2| = |A_2A_3| = |A_3A_4|$, $|A_0A_2| = |A_1A_3| = |A_2A_4|$, $|A_0A_3| = |A_1A_4|$. Then, cosines of the angles at vertices A_1, A_2 and A_3 are equal.
(Proof: With a CAS, via elimination and contradiction.)

Prop. 1 Given the points $A_0 = (0,0)$, $A_1 = (1,0)$, …, $A_4 = (x_4, y_4)$ in the plane, $|A_0A_1| = |A_1A_2| = |A_2A_3| = |A_3A_4|$, $|A_0A_2| = |A_1A_3| = |A_2A_4|$, $|A_0A_3| = |A_1A_4|$. Then, cosines of the angles at vertices A_1, A_2 and A_3 are equal.
(Proof: With a CAS, via elimination and contradiction.)

Prop. 2 Given the points $A_0 = (0,0)$, $A_1 = (1,0)$, …, $A_4 = (x_4, y_4)$ in the plane, $|A_0A_1| = |A_1A_2| = |A_2A_3| = |A_3A_4|$, $|A_0A_2| = |A_1A_3| = |A_2A_4|$, $|A_0A_3| = |A_1A_4|$. We assume that the oriented distance of A_2 from the line A_0A_1 is the same in absolute value but has a different sign as the oriented distance of A_3 from the line A_1A_2. Then, the resulting polyline $A_0A_1A_2A_3A_4$ forms a W-like shape or a straight polyline (unless $A_0 = A_4$).
(Proof: With a CAS + human thinking, geometrically.)

Proof of Prop. 1

First part: Angles at A_1 and A_2 are equal

```
X:=[0,1,x2,x3,x4]
Y:=[0,0,y2,y3,y4]
p:=[undef]
sqdist(i,j):=(X[j]-X[i])^2+(Y[j]-Y[i])^2
sprod(i,j,k):=(X[j]-X[i])*(X[j]-X[k])+(Y[j]-Y[i])*(Y[j]-Y[k])
l:=0
for(k=1;k<=3;k++) { for(j=1;j<=4-k;j++,l++)
↪  {p[l]:=sqdist(j,j+k)-sqdist(0,k)}}
p[l]:=t
p[l]:=p[l]*(sprod(0,1,2)-sprod(1,2,3))
p[l]:=p[l]-1
el1:=eliminate(p,[x3,y3,x4,y4,t])
// Elimination gives [1] which means success (via contradiction)
```

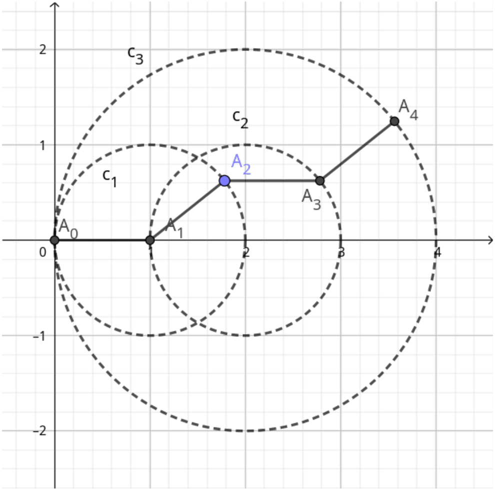
Proof of Prop. 2

Obtaining elimination ideals that can be analyzed geometrically

```
X:=[0,1,x2,x3,x4]
Y:=[0,0,y2,y3,y4]
p:=[undef]
sqdist(i,j):=(X[j]-X[i])^2+(Y[j]-Y[i])^2
sprod(i,j,k):=(X[j]-X[i])*(X[j]-X[k])+(Y[j]-Y[i])*(Y[j]-Y[k])
halfpl(i,j,k):=(X[k]-X[j])*(Y[i]-Y[j])-(X[i]-X[j])*(Y[k]-Y[j])
l:=0
for(k=1;k<=3;k++) { for(j=1;j<=4-k;j++, l++)
↪  {p[l]:=sqdist(j,j+k)-sqdist(0,k)}}
p[l]:=t
p[l]:=p[l]*(halfpl(0,1,2)-halfpl(1,2,3))
p[l]:=p[l]-1
el1:=eliminate(p,[x3,y3,x4,y4,t])
el2:=eliminate(p,[x2,y2,x4,y4,t])
el3:=eliminate(p,[x2,y2,x3,y3,t])
```

Proof of Prop. 2

Human-driven geometrical analysis



Eliminations yield the ideals $\langle x_2^2 + y_2^2 - 2x_2 \rangle$, $\langle x_3^2 + y_3^2 - 4x_3 + 3 \rangle$, $\langle x_4^2 + y_4^2 - 4x_4 \rangle$, shown as algebraic varieties.

Applications and future work

- See another talk “Proving theorems on regular polygons by elimination – the elementary way”, including theorems on the Golden ratio, paper folding (of a regular heptagon), a theorem on regular 9-gons (classroom applications).

Applications and future work

- See another talk "Proving theorems on regular polygons by elimination – the elementary way", including theorems on the Golden ratio, paper folding (of a regular heptagon), a theorem on regular 9-gons (classroom applications).
- This technique can be used to prove a conjecture of Paul Erdős (`https://arxiv.org/abs/2412.05190`, "A note on Erdős's mysterious remark", Z.K.).

Applications and future work

- See another talk “Proving theorems on regular polygons by elimination – the elementary way”, including theorems on the Golden ratio, paper folding (of a regular heptagon), a theorem on regular 9-gons (classroom applications).
- This technique can be used to prove a conjecture of Paul Erdős (`https://arxiv.org/abs/2412.05190`, “A note on Erdős’s mysterious remark”, Z.K.).
- Conjectures:
  - If $n \geq 3$, n is prime, $N_1 \wedge D_1 \wedge D_2$ is sufficient.
  - If $n \geq 6$, the assumptions N_1, N_d for each $d|n$, D_1, D_2 and D_3 are sufficient (that is, the non-degeneracy conditions can be minimized).
  - If $n \geq 3$, the assumptions N_1, N_d for each $d|n$, D_1, D_2, …, $D_{\lfloor \frac{n}{2} \rfloor}$ are sufficient (a weaker form).
  - Non-degeneracy conditions always help to avoid the issues with complex coordinates.

THANK YOU!

References

Z. Kovács.
Automated detection of interesting properties in regular polygons.
Mathematics in Computer Science, 14:727–755, 2020.

J. Rabinowitsch.
Zum Hilbertschen Nullstellensatz.
Mathematische Annalen, 102(1):520, 1929.

W. Watkins and J. Zeitlin.
The minimal polynomial of $\cos(2\pi/n)$.
The American Mathematical Monthly, 100(5):471–474, 1993.