Manifold-based Proving Methods in Projective Geometry

Michael Katzenberger and Jürgen Richter-Gebert
Technical University of Munich

August 1, 2025

Background

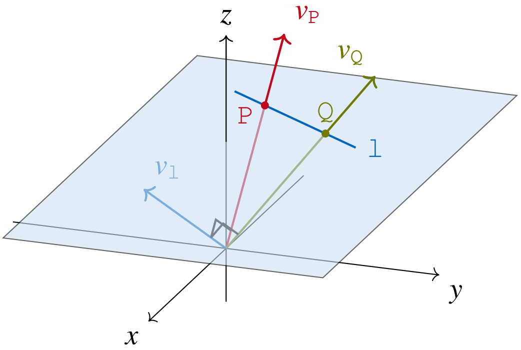
The interest in this topic arrises from 2 recent events:

- Desire to compare paper “Incidences and Tilings” by Sergey Fomin and Pavlo Pylyavskyy with Jürgen Richter-Gebert’s methods from 20 years ago
- New implementation of Dynamic Geometry Software “Cinderella” far enough that an automatic prover seems in reach

The Setting

Projective plane:

- Points $\mathtt{P}$ and lines $\mathtt{l}$ represented by homogeneous coordinates in $\mathbb{R}^3$
- $\mathtt{P}$ lies on $\mathtt{l} \Leftrightarrow \langle \mathtt{P}, \mathtt{l} \rangle = 0$
- 3 Points $\mathtt{P}, \mathtt{Q}, \mathtt{R}$ collinear $\Leftrightarrow [\mathtt{P}, \mathtt{Q}, \mathtt{R}] := \det(\mathtt{P}, \mathtt{Q}, \mathtt{R}) = 0$



Projective Incidence Theorems

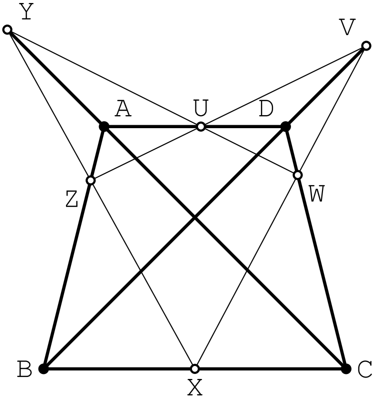
Certain incidences hold

$+$

Certain incidences do not hold

$\Downarrow$

A new incidence holds



Not a restriction, as Euclidean properties can be expressed as projective properties

Binomial Proofs

For any $\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{E} \in \mathbb{R}^3$ the following equation holds (Grassman-Plücker-Relation):

$$[\mathtt{A},\mathtt{B},\mathtt{C}][\mathtt{A},\mathtt{D},\mathtt{E}] - [\mathtt{A},\mathtt{B},\mathtt{D}][\mathtt{A},\mathtt{C},\mathtt{E}] + [\mathtt{A},\mathtt{B},\mathtt{E}][\mathtt{A},\mathtt{C},\mathtt{D}] = 0$$

This means, if $[\mathtt{A},\mathtt{D},\mathtt{E}] \neq 0$ gilt:

$$[\mathtt{A},\mathtt{B},\mathtt{C}] = 0 \Longleftrightarrow \frac{[\mathtt{A},\mathtt{B},\mathtt{D}][\mathtt{A},\mathtt{C},\mathtt{E}]}{[\mathtt{A},\mathtt{B},\mathtt{E}][\mathtt{A},\mathtt{C},\mathtt{D}]} = 1$$

Binomial Proofs

$$\textit{Each collinearity generates a fraction } \frac{[\mathtt{A},\mathtt{B},\mathtt{D}][\mathtt{A},\mathtt{C},\mathtt{E}]}{[\mathtt{A},\mathtt{B},\mathtt{E}][\mathtt{A},\mathtt{C},\mathtt{D}]} = 1$$

$$+$$

$$\textit{All appearing determinants are } \neq 0$$

$$\Downarrow$$

The product of all fractions becomes a fraction of the same form

and thus implies a new collinearity

Other properties like 4 points on a circle, 6 points on a conic or orthogonality of lines can also be encoded to similar fractions being equal to one

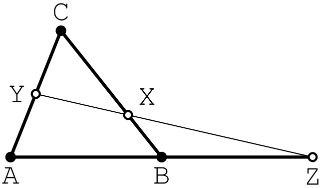
Binomial Proof for Desargues Theorem

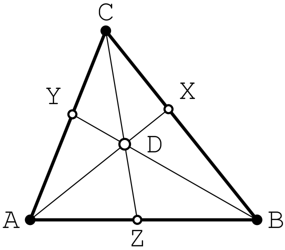
$$
\begin{array}{lcl}
\mathtt{A,B,Z}\ \text{collinear} & \Longrightarrow & [\mathtt{Z,A,U}][\mathtt{Z,B,X}] = [\mathtt{Z,A,X}][\mathtt{Z,B,U}] \\
\mathtt{B,C,X}\ \text{collinear} & \Longrightarrow & [\mathtt{C,X,Y}][\mathtt{B,X,V}] = [\mathtt{C,X,V}][\mathtt{B,X,Y}] \\
\mathtt{C,D,W}\ \text{collinear} & \Longrightarrow & [\mathtt{W,C,X}][\mathtt{W,D,U}] = [\mathtt{W,C,U}][\mathtt{W,D,X}] \\
\mathtt{D,A,U}\ \text{collinear} & \Longrightarrow & [\mathtt{D,U,V}][\mathtt{A,U,Y}] = [\mathtt{D,U,Y}][\mathtt{A,U,V}] \\
\mathtt{B,D,V}\ \text{collinear} & \Longrightarrow & [\mathtt{B,V,U}][\mathtt{D,V,X}] = [\mathtt{B,V,X}][\mathtt{D,V,U}] \\
\mathtt{U,Z,V}\ \text{collinear} & \Longrightarrow & [\mathtt{U,V,A}][\mathtt{U,Z,B}] = [\mathtt{U,V,B}][\mathtt{U,Z,A}] \\
\mathtt{X,W,V}\ \text{collinear} & \Longrightarrow & [\mathtt{X,V,C}][\mathtt{X,W,D}] = [\mathtt{X,V,D}][\mathtt{X,W,C}] \\
\mathtt{A,C,Y}\ \text{collinear} & \Longrightarrow & [\mathtt{A,Y,X}][\mathtt{C,Y,U}] = [\mathtt{A,Y,U}][\mathtt{C,Y,X}] \\
\mathtt{U,W,Y}\ \text{collinear} & \Longrightarrow & [\mathtt{U,Y,D}][\mathtt{U,W,C}] = [\mathtt{U,Y,C}][\mathtt{U,W,D}]
\end{array}
$$

$$\Downarrow$$

$$[\mathtt{X,Y,B}][\mathtt{X,Z,A}] = [\mathtt{X,Y,A}][\mathtt{X,Z,B}] \quad \Longrightarrow \quad \mathtt{X,Y,Z}\ \text{collinear}$$

Menelaus' und Ceva's Theorems

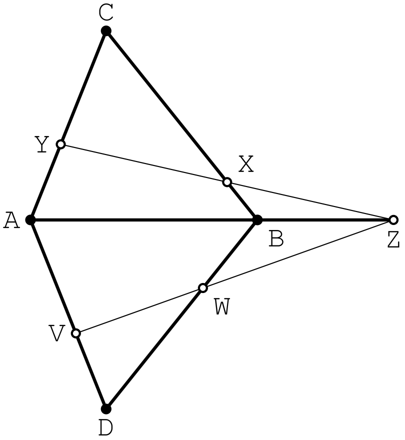




$$\frac{|\overrightarrow{AZ}|}{|\overrightarrow{ZB}|} \cdot \frac{|\overrightarrow{BX}|}{|\overrightarrow{XC}|} \cdot \frac{|\overrightarrow{CY}|}{|\overrightarrow{YA}|} = -1$$

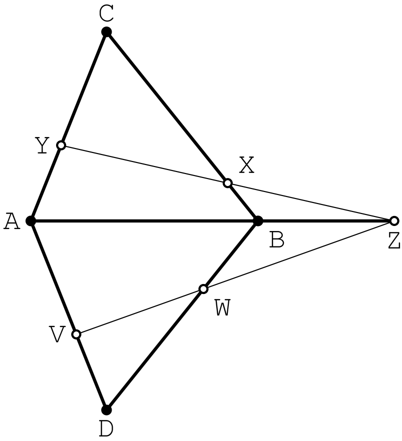
$$\frac{|\overrightarrow{AZ}|}{|\overrightarrow{ZB}|} \cdot \frac{|\overrightarrow{BX}|}{|\overrightarrow{XC}|} \cdot \frac{|\overrightarrow{CY}|}{|\overrightarrow{YA}|} = 1$$

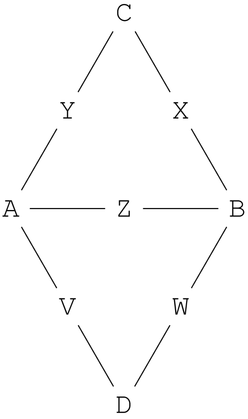
The Gluing-Process



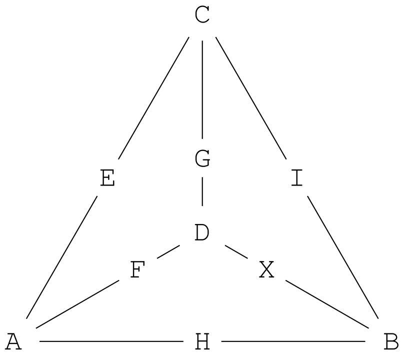
$$
\begin{aligned}
&\frac{\overrightarrow{|BX|}}{\overrightarrow{|XC|}} \cdot \frac{\overrightarrow{|CY|}}{\overrightarrow{|YA|}} \cdot \frac{\overrightarrow{|AV|}}{\overrightarrow{|VD|}} \cdot \frac{\overrightarrow{|DW|}}{\overrightarrow{|WB|}} \\
= &\frac{\overrightarrow{|AZ|}}{\overrightarrow{|ZB|}} \cdot \frac{\overrightarrow{|BX|}}{\overrightarrow{|XC|}} \cdot \frac{\overrightarrow{|CY|}}{\overrightarrow{|YA|}} \\
\cdot &\frac{\overrightarrow{|AV|}}{\overrightarrow{|VD|}} \cdot \frac{\overrightarrow{|DW|}}{\overrightarrow{|WB|}} \cdot \frac{\overrightarrow{|BZ|}}{\overrightarrow{|ZA|}} \\
= &(-1) \cdot (-1) = 1
\end{aligned}
$$

The Gluing-Process

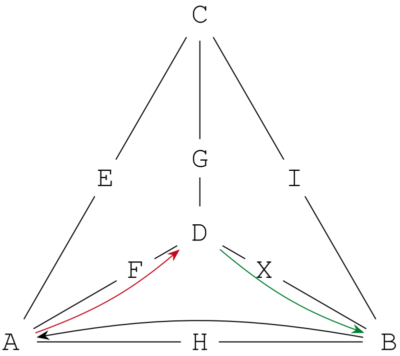




The Gluing-Process

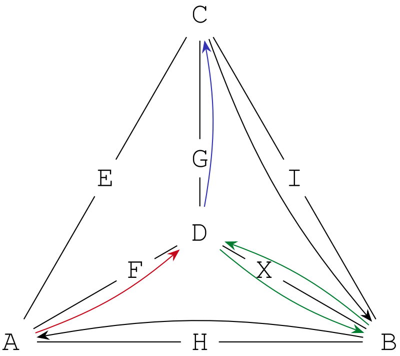


The Gluing-Process



$$\frac{\overrightarrow{|AF|}}{\overrightarrow{|FD|}} \cdot \frac{\overrightarrow{|DX|}}{\overrightarrow{|XB|}} \cdot \frac{\overrightarrow{|BH|}}{\overrightarrow{|HA|}}$$

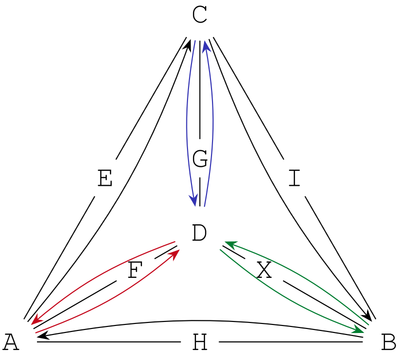
The Gluing-Process



$$\frac{\overrightarrow{|AF|}}{\overrightarrow{|FD|}} \cdot \frac{\overrightarrow{|DX|}}{\overrightarrow{|XB|}} \cdot \frac{\overrightarrow{|BH|}}{\overrightarrow{|HA|}}$$

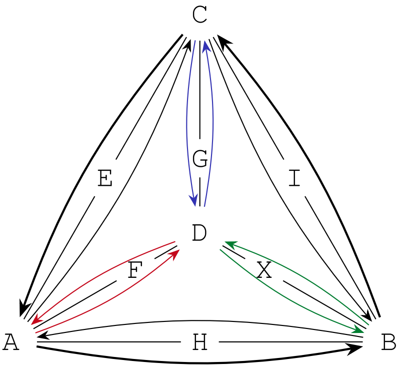
$$\cdot \frac{\overrightarrow{|BX|}}{\overrightarrow{|XD|}} \cdot \frac{\overrightarrow{|DG|}}{\overrightarrow{|GC|}} \cdot \frac{\overrightarrow{|CI|}}{\overrightarrow{|IB|}}$$

The Gluing-Process



$$\frac{\overrightarrow{|AF|}}{\overrightarrow{|FD|}} \cdot \frac{\overrightarrow{|DX|}}{\overrightarrow{|XB|}} \cdot \frac{\overrightarrow{|BH|}}{\overrightarrow{|HA|}} \cdot \frac{\overrightarrow{|BX|}}{\overrightarrow{|XD|}} \cdot \frac{\overrightarrow{|DG|}}{\overrightarrow{|GC|}} \cdot \frac{\overrightarrow{|CI|}}{\overrightarrow{|IB|}} \cdot \frac{\overrightarrow{|CG|}}{\overrightarrow{|GD|}} \cdot \frac{\overrightarrow{|DF|}}{\overrightarrow{|FA|}} \cdot \frac{\overrightarrow{|AE|}}{\overrightarrow{|EC|}} = -1$$

The Gluing-Process



$$\frac{\overrightarrow{|\mathtt{AF}|}}{\overrightarrow{|\mathtt{FD}|}} \cdot \frac{\overrightarrow{|\mathtt{DX}|}}{\overrightarrow{|\mathtt{XB}|}} \cdot \frac{\overrightarrow{|\mathtt{BH}|}}{\overrightarrow{|\mathtt{HA}|}}$$

$$\cdot \frac{\overrightarrow{|\mathtt{BX}|}}{\overrightarrow{|\mathtt{XD}|}} \cdot \frac{\overrightarrow{|\mathtt{DG}|}}{\overrightarrow{|\mathtt{GC}|}} \cdot \frac{\overrightarrow{|\mathtt{CI}|}}{\overrightarrow{|\mathtt{IB}|}}$$

$$\cdot \frac{\overrightarrow{|\mathtt{CG}|}}{\overrightarrow{|\mathtt{GD}|}} \cdot \frac{\overrightarrow{|\mathtt{DF}|}}{\overrightarrow{|\mathtt{FA}|}} \cdot \frac{\overrightarrow{|\mathtt{AE}|}}{\overrightarrow{|\mathtt{EC}|}} = -1$$

$$\Rightarrow \frac{\overrightarrow{|\mathtt{AH}|}}{\overrightarrow{|\mathtt{HB}|}} \cdot \frac{\overrightarrow{|\mathtt{BI}|}}{\overrightarrow{|\mathtt{IC}|}} \cdot \frac{\overrightarrow{|\mathtt{CE}|}}{\overrightarrow{|\mathtt{EA}|}} = -1$$

Ceva-Menelaus-Proofs

Triangulation of a closed oriented Surface

$$+$$

All triangles except one are Ceva-/Menelaus-triangles

$$\Downarrow$$

The last triangle must be a Ceva-/Menelaus-triangle

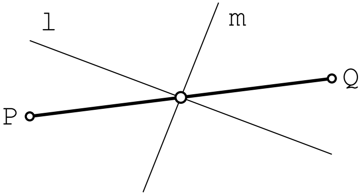
Equivalence to Binomial Proofs

Theorem (Jürgen Richter-Gebert, Susanne Apel, 2009)

Under the assumption of being able to add two generic points, A projective incidence theorem has a Ceva-Menelaus-proof if, and only if, it has a binomial proof.

Proofs using Quadrilateral Tilings (Fomin & Pylyavskyy)

$$\begin{tikzcd} \mathtt{P} \arrow[d] & \mathtt{l} \arrow[l] \\ \mathtt{m} \arrow[r] & \mathtt{Q} \arrow[u] \end{tikzcd}$$



coherent, if $\quad \dfrac{\langle \mathtt{P}, \mathtt{l} \rangle \cdot \langle \mathtt{Q}, \mathtt{m} \rangle}{\langle \mathtt{P}, \mathtt{m} \rangle \cdot \langle \mathtt{Q}, \mathtt{l} \rangle} = 1$

Proofs using Quadrilateral Tilings (Fomin & Pylyavskyy)

Qudrilateral Tiling of a closed oriented surface

$$+$$

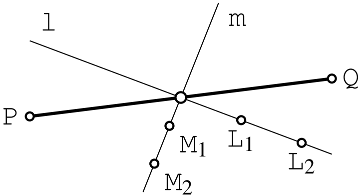
All quadrangles but one are coherent

$$\Downarrow$$

The last quadrangle must be coherent

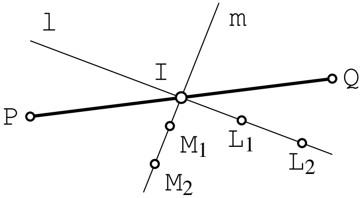
Relation to Binomial Proofs

$$\begin{tikzcd} \mathtt{P} \arrow[d] & \mathtt{l} \arrow[l] \\ \mathtt{m} \arrow[r] & \mathtt{Q} \arrow[u] \end{tikzcd}$$



$$\frac{\langle \mathtt{P}, \mathtt{l}\rangle \cdot \langle \mathtt{Q}, \mathtt{m}\rangle}{\langle \mathtt{P}, \mathtt{m}\rangle \cdot \langle \mathtt{Q}, \mathtt{l}\rangle} = 1 \iff \frac{[\mathtt{P}, \mathtt{L}_1, \mathtt{L}_2][\mathtt{Q}, \mathtt{M}_1, \mathtt{M}_2]}{[\mathtt{P}, \mathtt{M}_1, \mathtt{M}_2][\mathtt{Q}, \mathtt{L}_1, \mathtt{L}_2]} = 1$$

Relation to Binomial Proofs



$$\frac{[\mathtt{P},\mathtt{L}_1,\mathtt{L}_2][\mathtt{Q},\mathtt{M}_1,\mathtt{M}_2]}{[\mathtt{P},\mathtt{M}_1,\mathtt{M}_2][\mathtt{Q},\mathtt{L}_1,\mathtt{L}_2]}$$

$$= \underbrace{\frac{[\mathtt{L}_1,\mathtt{L}_2,\mathtt{P}][\mathtt{L}_1,\mathtt{I},\mathtt{Q}]}{[\mathtt{L}_1,\mathtt{L}_2,\mathtt{Q}][\mathtt{L}_1,\mathtt{I},\mathtt{P}]}}_{\text{GPR for } [\mathtt{L}_1,\mathtt{L}_2,\mathtt{I}]=0} \cdot \underbrace{\frac{[\mathtt{I},\mathtt{P},\mathtt{L}_1][\mathtt{I},\mathtt{Q},\mathtt{M}_1]}{[\mathtt{I},\mathtt{P},\mathtt{M}_1][\mathtt{I},\mathtt{Q},\mathtt{L}_1]}}_{\text{GPR for } [\mathtt{I},\mathtt{P},\mathtt{Q}]=0} \cdot \underbrace{\frac{[\mathtt{M}_1,\mathtt{I},\mathtt{P}][\mathtt{M}_1,\mathtt{M}_2,\mathtt{Q}]}{[\mathtt{M}_1,\mathtt{I},\mathtt{Q}][\mathtt{M}_1,\mathtt{M}_2,\mathtt{P}]}}_{\text{GPR for } [\mathtt{M}_1,\mathtt{M}_2,\mathtt{I}]=0}$$

Relation to Binomial Proofs

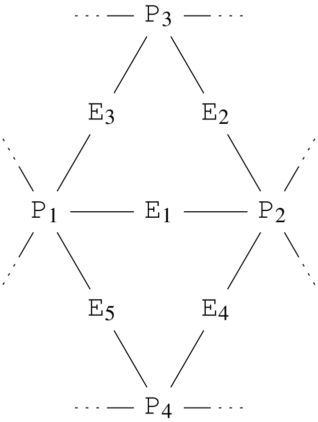
Theorem

If a projective incidence theorem has a quad proof, then it also has a binomial proof.

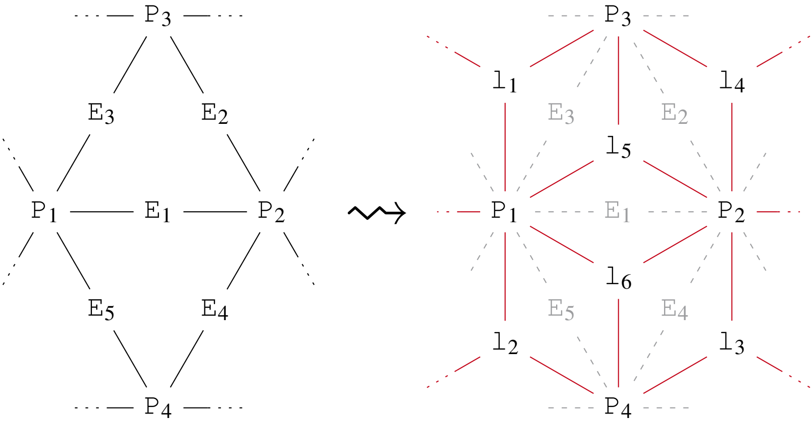
Hierarchy of Manifold-Based Proving Methods

$$
\begin{array}{ccc}
B & \Longleftrightarrow & CM \\
\Downarrow & & \Downarrow \\
F & \Longleftrightarrow & M \\
 & & \Downarrow \\
 & & C
\end{array}
$$

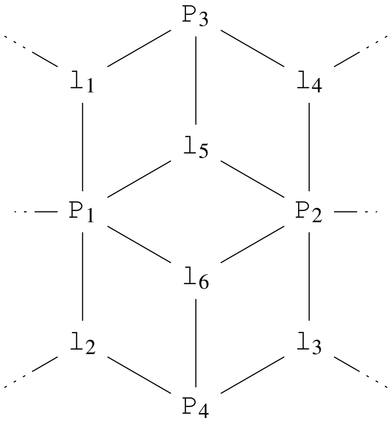
Relation to Ceva-Menelaus-Proofs



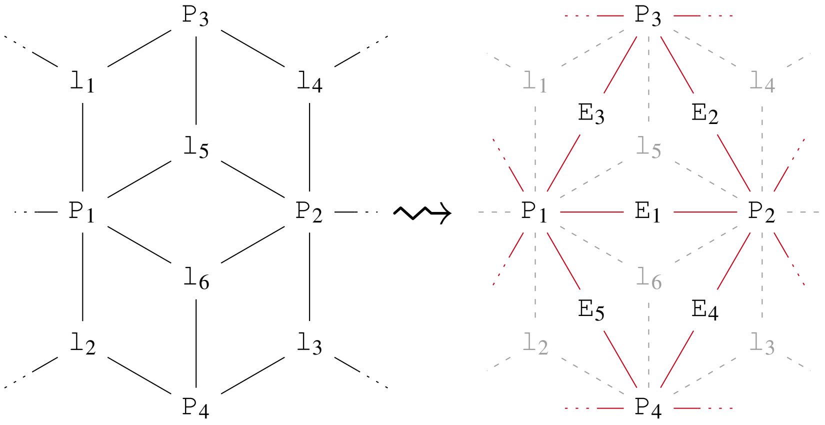
Relation to Ceva-Menelaus-Proofs



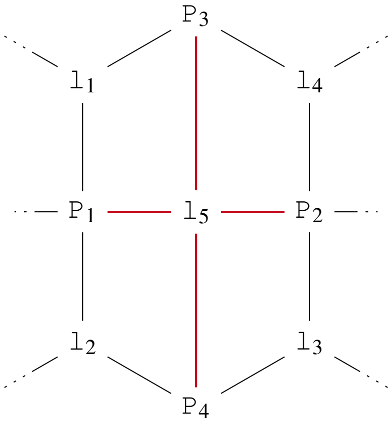
Relation to Ceva-Menelaus-Proofs



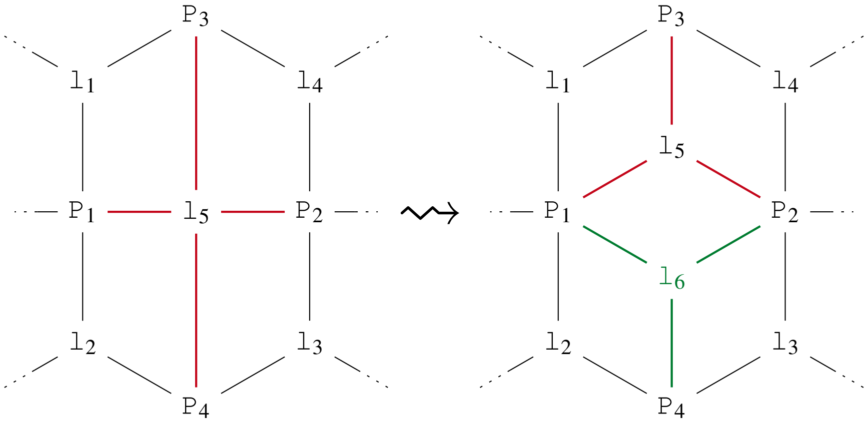
Relation to Ceva-Menelaus-Proofs



Relation to Ceva-Menelaus-Proofs



Relation to Ceva-Menelaus-Proofs

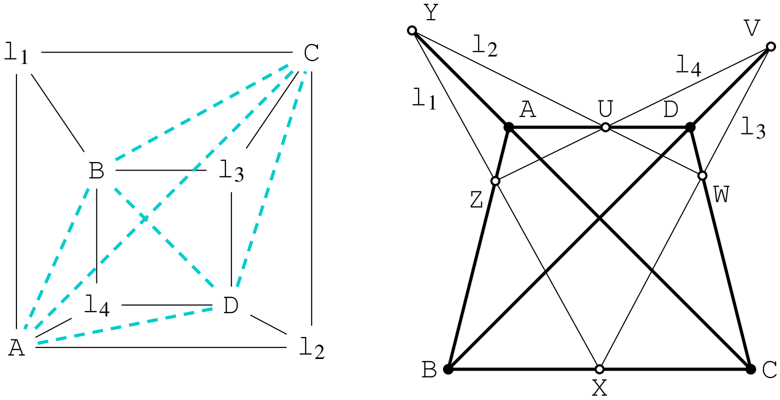


Relation to Ceva-Menelaus-Proofs

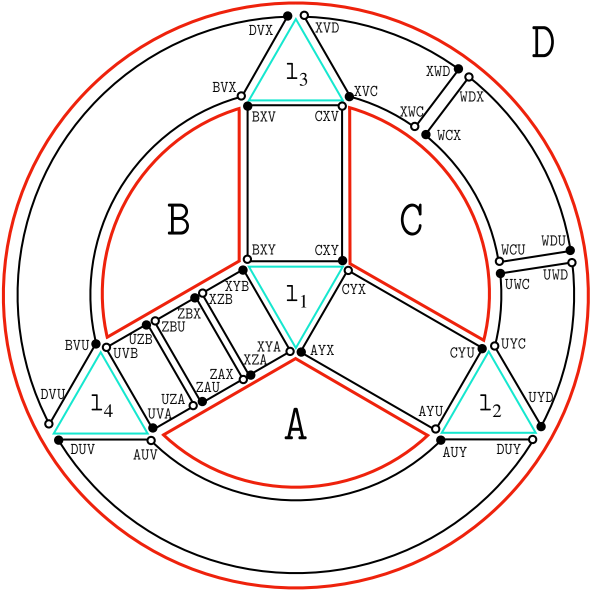
Theorem

A projective incidence theorem has a quad-proof if, and only if, it also has a Ceva-Menelaus-proof containing only Menelaus-triangles (pure Menelaus-proof).

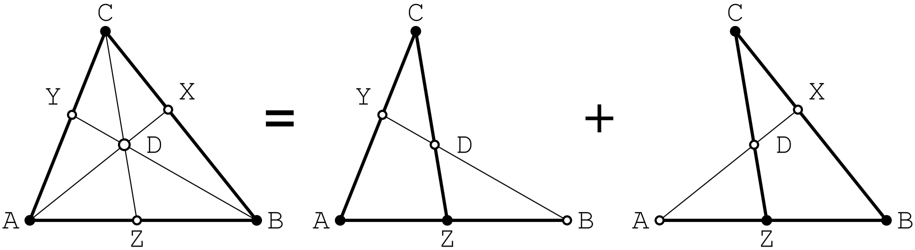
An Illustrative Example



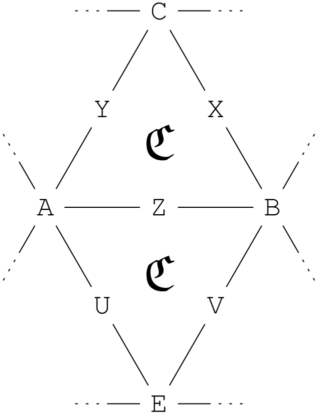
An Illustrative Example



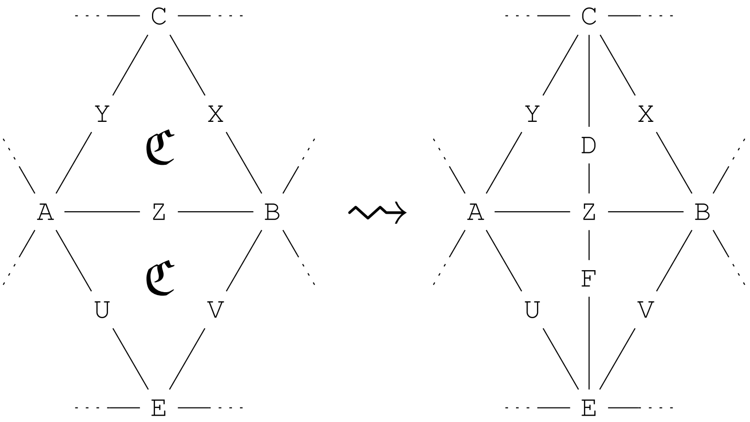
What about Pure Ceva-Proofs?



What about Pure Ceva-Proofs?



What about Pure Ceva-Proofs?



What about Pure Ceva-Proofs?

Theorem

Each projective incidence theorem with a Ceva-Menelaus-proof containing exclusively pairwise appearing Ceva-triangles, also has a pure Menelaus-Beweis.

In particular, each projective incidence theorem with a pure Ceva-proof has a pure Menelaus-proof.