Towards a Structural Analysis of Interesting Geometric Theorems: Analysis of a Survey

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Interesting Geometric Theorems

The concept of what makes a geometric theorem "interesting" is deeply rooted in human intuition, pedagogy, and aesthetic perception. Despite its centrality in the mathematical experience, "interestingness" remains elusive to formal characterisation.

In our project, we aim to present a methodology capable of exploring the structural properties of geometric theorems that may underlie their perceived interestingness. Our approach systematically integrates human-based survey data, automated theorem proving, and Geometrographic[1] analysis [6, 7, 8, 10].

[1] Geometrography, "alias the art of geometric constructions", aims at providing a tool: (i) to designate every geometric construction by a symbol that manifests its simplicity and exactitude;. . .

First Step — Survey

First, we will show how it is possible to build upon the results of a comprehensive survey conducted among mathematicians, educators, and students.[2]

Participants were asked to list geometric theorems they considered interesting and to provide qualitative explanations for their choices.

This phase allowed us to capture the human dimension of mathematical interest, revealing subjective factors such as simplicity, elegance, surprise, utility, and conceptual depth.

[2] Mostly University professors and students (pre-service high-school teachers)

Second Step — Formal Proofs

Second, we will show how to construct formal proofs for each identified theorem using automated theorem proving tools, specifically employing the Area Method [4], (and similar methods that can be characterised geometrographically) as implemented in the GCLC prover and/or provers based on the deductive database method [1, 2, 3].

These proofs will be synthetically generated, ensuring uniformity and enabling precise structural analysis.

Third Step — Geometrographic coefficients

Third, we will show how the given formal proof can be analysed using a set of quantitative metrics, known as Geometrographic [5, 9] coefficients, including [6, 8].

- $\mathrm{CS}_{\mathrm{proof}}$, coefficient of simplicity giving the simplicity coefficient for the overall proof;
- $\mathrm{CS}_{\mathrm{gcl}}$, coefficient of simplicity for the geometric construction (the conjecture);
- $\mathrm{CT}_{\mathrm{proof}}$, the total number of steps in the proof;
- $\mathrm{CS}_{\mathrm{proofmax}}$, the highest simplicity coefficient of the lemmas/definitions applications, it gives the simplicity coefficient for the most difficult step of the proof;
- $\mathrm{CD}_{\mathrm{typeproof}}$, the number of different types of lemmas used in the proof;
- $\mathrm{CD}_{\mathrm{highproof}}$, the number of different steps of high difficulty in the proof.

GRCP, the Geometrographic Readability Coefficient of the Proof [6].

Additionally, proof traces (line charts, mapping step difficulty) are generated to visually assess the complexity progression.

Research Question

Our central research objective is to investigate whether there exist structural features in proofs that statistically correlate with the human perception of interestingness.

Analysis of the Survey Responses

Analysis of the Survey Responses: "Interesting Geometric Theorems"

We start a survey, open to all the mathematic community, late 2023 (first answers dates from 2023/07/21) and is an ongoing effort (the last, for now, answer is from 2025/05/31). To date we count 21 answers.

Survey

The collected data, mainly qualitative, provide an overview of personal experiences and the characteristics attributed to the theorems considered interesting.

Situations in Which the Adjective “Interesting” Was Used

The most common contexts include:

- Lessons or university courses: many participants recalled moments during formal education when a theorem particularly captured their interest.
- Personal discoveries: some mentioned self-study experiences or moments of personal intuition.
- Practical applications: situations where a theorem had a concrete application or helped solve a specific problem.

Geometric Theorems Considered Interesting

Among the most frequently mentioned:

- Pythagorean Theorem: Appreciated for its simplicity and wide range of applications.
- Thales' Theorem: Recognized for its elegance and historical importance.
- Theorem of the Three Perpendiculars: Noted for its structure and implications.
- Hjelmslev's Theorem: Considered interesting for its unique geometric properties.
- Fermat's Last Theorem: Cited for its fascinating history and the complexity of its proof.

Characteristics that make a Theorem "Interesting"

- Simplicity and elegance: Theorems with clear statements and concise proofs.
- Applicability: Theorems that are useful in multiple contexts or solve practical problems.
- Surprise or counterintuitiveness: Theorems that yield unexpected results or challenge intuition.
- Conceptual depth: Theorems that offer deeper understanding of geometry or connect different areas of mathematics.
- Historical significance: Theorems that have played a major role in the history of mathematics.

Operational Definition

An interesting theorem in geometry is a statement that exhibits at least some of the following characteristics:

- Simplicity and clarity: the theorem has a statement that is simple to understand and a proof that, even if deep, remains accessible or elegant.
- Capacity to surprise: it leads to an unexpected or counterintuitive result, or it connects geometric elements in a new or surprising way.
- Conceptual depth: it reveals hidden relationships or allows different concepts in geometry (or in broader mathematics) to be connected.
- Usefulness and applicability: it is useful for solving practical problems or geometric constructions, or it finds applications in other scientific or educational contexts.
- Aesthetic value: it is appreciated for its formal "beauty", its symmetry, or the elegance of the ideas involved.
- Historical or emotional context: it is linked to important historical discoveries or is remembered because it is associated with formative moments, personal insights, or memorable teaching experiences.

Practical Criteria to Recognize an Interesting Theorem

To assess whether a geometric theorem is interesting, you can ask:

- Is it easy to state but deep to understand?
- Does it lead to a result that I would not have intuitively expected?
- Does it help me solve problems or build new knowledge?
- Does it strike me with its beauty, elegance, or power?
- Is it connected to major discoveries or has it changed how an area of mathematics is viewed?
- Is it a theorem I vividly remember from study or personal experience?

If at least two or more of these questions are answered "yes", then the theorem can be considered operationally interesting.

List of Interesting Theorems

Based on the 20 survey responses

- A tetrahedron can be inscribed into a sphere
- Angle Bisector Theorem
- Brianchon's Theorem
- Ceva's Theorem
- Cocircularity of midpoints and feet of a triangle
- Concurrence of notable lines in a triangle
- . . .

Geometrographic Correspondence

- Simplicity and Elegance

  These traits correspond to low values in the simplicity coefficient of the proof $\mathrm{CS}_{\mathrm{proof}}$, the maximum difficulty of any single step ($\mathrm{CS}_{\mathrm{proofmax}}$), and an overall low GRCP. For example, Ceva's Theorem has a $\mathrm{CS}_{\mathrm{proof}}$ of 220 and a GRCP of 564, making it both highly readable, and subjectively interesting.

- Capacity to Surprise

  This can be reflected in the proof trace (line chart), where sudden jumps in step complexity may indicate conceptually surprising moments. High isolated $\mathrm{CS}_{\mathrm{proofmax}}$ values also point to these potential "aha!" moments.

Geometrographic Correspondence

- **Conceptual Depth**

  Such proofs often involve multiple types of lemmas ($\mathrm{CD}_{\mathrm{typeproof}}$) and may include a few high-difficulty steps ($\mathrm{CD}_{\mathrm{highproof}}$). The overall simplicity might not be low, but the balance of structure and richness results in a moderate GRCP. This is the case with the Circumcenter Theorem, which has a $\mathrm{CS}_{\mathrm{proof}}$ of 8554 and a GRCP of 127 408 — classified as medium readability.

- **Usefulness and Applicability**

  Theorems that are involved in constructions with low geometric complexity ($\mathbf{CS}_{\mathbf{gcl}}$) or appear reusable may reflect a structural simplicity that aligns with this sense of usefulness.

- **Historical and Emotional Context**

  While this type of context is not directly quantifiable in the formal coefficients, it's possible to compare frequently cited theorems from the survey with their GRCP values to see whether "interesting" theorems are also formally "readable".

Interestingness Issue and Future Works

Having already work on the subject of readability of proofs [6, 8] linking the computational and the human sides, the authors want to continue pursuing in finding an answer, even if partial, to Larry Wos 31^{st} problem [10].

31^{st} **Problem:**

What properties can be identified to permit an automated reasoning program to find new and interesting theorems, as opposed to proving conjectured theorems?

Interestingness Issue and Future Works

Moreover, the connection between subjective perception and formal readability opens several avenues for research:

- Cross-analyse survey responses (e.g., Ceva, Thales, Pythagoras) with GRCP values from the Geometrography model.

[3] Thousands of Geometric problems for Theorem Provers, http://hilbert.mat.uc.pt/TGTP/

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- Add survey-derived metadata to repositories like TGTP[3] enabling further study on how mathematical interest aligns with formal complexity.

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Conclusion

Our survey is an ongoing project, always open to receive new answers. We invite everyone to think about this subject and to answer (if not already done) to the survey.

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Survey

Thank You

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