Different Types of Locus Dependencies in Solving Geometry Construction Problems

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ADG, Stuttgart, 2025

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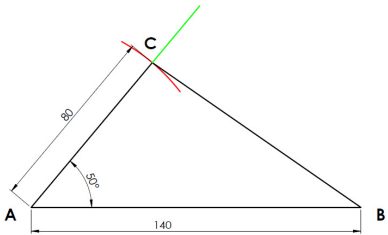
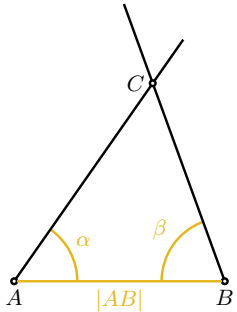
Geometric construction problems

- Well known mathematical problem with a long history and many practical applications
- Objective: Construct a geometric figure that meet specified constraints
- Different tools can be available – we focus on straightedge and compass construction
- Special attention is paid to triangle constructions:

Construct $\triangle ABC$ *given* $|AB|$, $|AC|$, *and* α

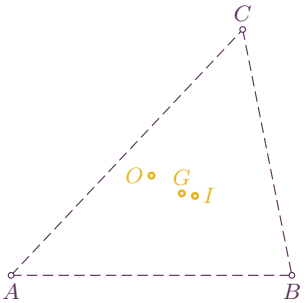


Types of construction problems



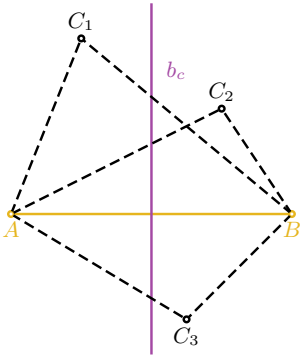
Solvable problem: α, β, $|AB|$

Types of construction problems



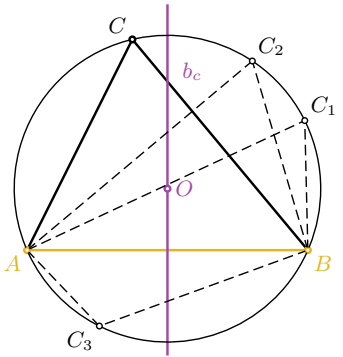
Unsolvable problem: O (circumcenter), G (centroid), I (incenter)

Types of construction problems



Redundant problem: A, B, b_c (bisector of $|AB|$)

Types of construction problems



Locus dependent (LD) problem: A, B, O (circumcenter)

Automating triangle constructions

- ArgoTriCS – system for automated solving of construction problems from the given corpus
- Focused on problems of constructing triangle that meets three given constraints
- Relies on background geometrical knowledge
- Automatically exports textual descriptions of constructions, as well as formal procedures in GCLC and JSON format

Motivation

- An emphasis is often, if not exclusively, put on problems known to be solvable
- Underdetermined problems (redundant and locus dependent ones) are rarely discussed
- LD problems play important role in both education, as well as in CAD and robotics
- In this work we focus on adapting ArgoTriCS to identify and solve LD problems
- We propose a new corpus of location construction problems suitable for exploring LD problems
- Dynamic visualization of all considered problems using our ArgoDG library is provided

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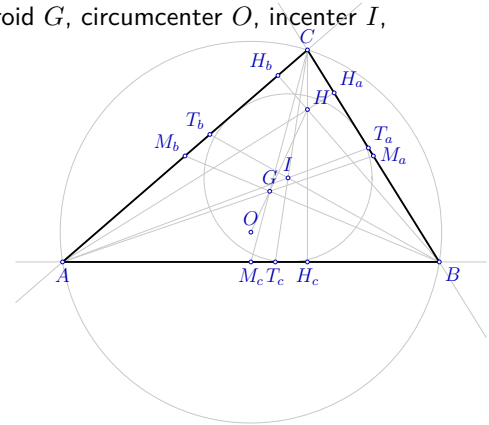
Corpora of construction problems

- Wernick's corpus (1982)
  - vertices A, B, C,
  - side midpoints M_a, M_b, M_c,
  - feet of altitudes H_a, H_b, H_c,
  - feet of internal angle bisectors T_a, T_b, T_c.
  - orthocenter H, centroid G, circumcenter O, incenter I,

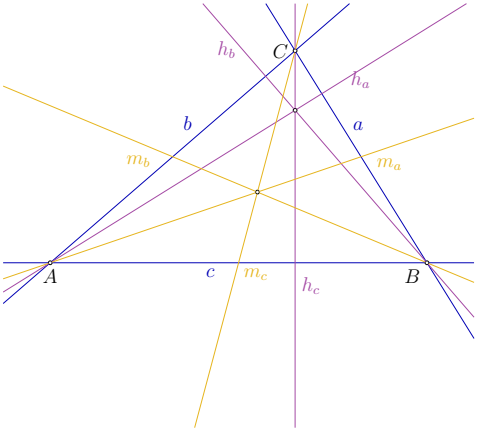


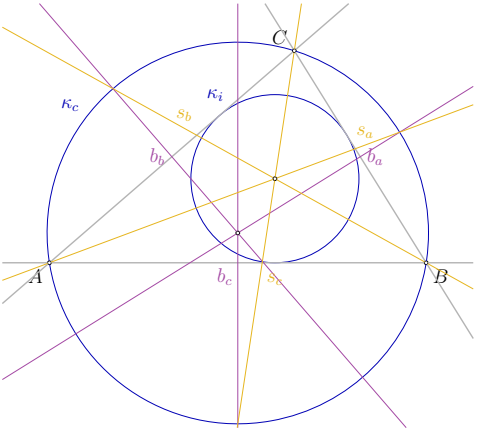
Corpora of construction problems

- Wernick's corpus (1982)
- Dual Triangle Construction Problem corpus (2025)
  - triangle sides a, b, c,
  - side bisectors b_a, b_b, b_c,
  - medians m_a, m_b, m_c,
  - altitudes h_a, h_b, h_c,
  - internal angle bisectors s_a, s_b, s_c,
  - circumcircle κ_c and incircle κ_i.

DTCP corpus

- Task: Construct $\triangle ABC$ given locations of three significant lines or circles associated with it





Adapting ArgoTriCS to accommodate a new corpus of problems

- Adding new primitive constructions:

Adapting ArgoTriCS to accommodate a new corpus of problems

- Adding new primitive constructions:
  1. Given a point P and a line l, one can construct a reflection of point P wrt. line l
  2. Given lines m and l, one can construct a reflection of line m wrt. line l
  3. Given a point P and a line l, one can construct a reflection of line l wrt. point P

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  3. Given a point P and a line l, one can construct a reflection of line l wrt. point P
  4. Given a circle κ and a line l, one can construct two lines that are tangent to κ and perpendicular to l
  5. Given a circle κ and its tangent l, one can construct the point at which they touch

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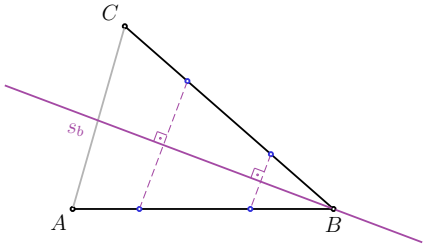
  5. Given a circle κ and its tangent l, one can construct the point at which they touch
  6. Given lines p, q, r that form a harmonic set of lines with a line s, one can construct the line s

Adapting ArgoTriCS to accommodate a new corpus of problems

- Adding new primitive constructions
- Introducing new lemmas:
  1. The vertex A lies on the reflection of an arbitrary line through the vertex B wrt. the midpoint of the side AB
  2. The reflection of an arbitrary point on the side BC wrt. the internal angle bisector s_b lies on the side AB



Adapting ArgoTriCS to accommodate a new corpus of problems

- Add new primitive constructions
- Introducing new lemmas:
  4. If for a line l it holds that triangle sides b, c, median m_a and the line l are a harmonic set of lines, i.e. $H(b, c, m_a, l)$, then the line l is parallel to the triangle side a
  5. If for a line l it holds that triangle medians m_b, m_c, m_a and the line l are a harmonic set of lines. i.e. $H(m_b, m_c, m_a, l)$, then the line l is parallel to the triangle side a

Adapting ArgoTriCS to accommodate a new corpus of problems

- Adding new primitive constructions
- Introducing new lemmas
- Prioritize constructions involving lines and circles

Adapting ArgoTriCS to accommodate a new corpus of problems

- Adding new primitive constructions
- Introducing new lemmas
- Prioritize constructions involving lines and circles
- The search algorithm remains the same

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Solving procedure

1. Is the problem redundant, i.e. can one of the given elements be constructed from the remaining two?
   - If yes, choose freely one of the vertices (not yet constructed) and proceed the search in a standard manner
   - If not, go to the next step

Solving procedure

1. Is the problem redundant, i.e. can one of the given elements be constructed from the remaining two?
2. Is the problem locus-dependent, i.e. does one of the given elements belong to some locus of points determined by the other two?
   - If yes, the dependent element is chosen arbitrarily, ensuring that it belongs to the specified locus. The search proceeds in a standard manner
   - If not, go to the next step

Solving procedure

1. Is the problem redundant, i.e. can one of the given elements be constructed from the remaining two?
2. Is the problem locus-dependent, i.e. does one of the given elements belong to some locus of points determined by the other two?
3. Can all three vertices of the triangle be constructed from the given elements?
   - If yes, the search terminates and the problem is considered solvable

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3. Can all three vertices of the triangle be constructed from the given elements?
   - If yes, the search terminates and the problem is considered solvable
   - If not and there are no primitive construction that can be applied, the problem is declared unsolvable using the given knowledge base

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Locus-dependent problems

- A solution to locus-dependent problem is possible only if specific conditions are met
- One of the given elements must lie on a locus defined by the others, with conditions like
  - two lines must be parallel
  - two lines must be perpendicular
  - point must lie on line/circle
  - center of circle must lie on the line
  - line has to be tangent to the circle, etc.
- These conditions are inferred from the knowledge embedded in ArgoTriCS

Types of locus-dependent problems

Problem with no solutions:

If the dependent element does not lie on the specified locus, the problem has no solution

Example 1

Construct a triangle ABC given its three altitudes which are not concurrent

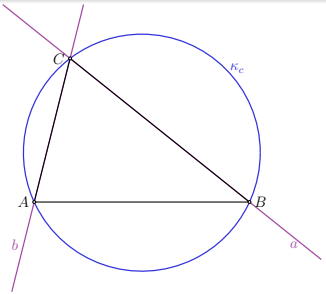
Types of locus-dependent problems

Problem with finitely many solutions:

If the dependent element lies on the specific locus and once appropriately chosen the construction proceeds as though the problem were solvable, the problem has finitely many solutions

Example 2

Construct a triangle ABC given its two sides a and b and circumcircle κ_c which are concurrent



Solution: a, b, κ_c

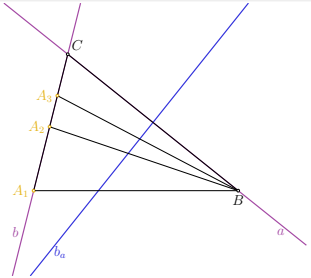
Types of locus-dependent problems

Problem with infinitely many solutions:

If the dependent element lies on the specific locus and once appropriately chosen the solution is not uniquely determined, the problem has infinitely many solutions

Example 3

Construct a triangle ABC given its sides a and b and the side bisector b_a, which is perpendicular to the line a

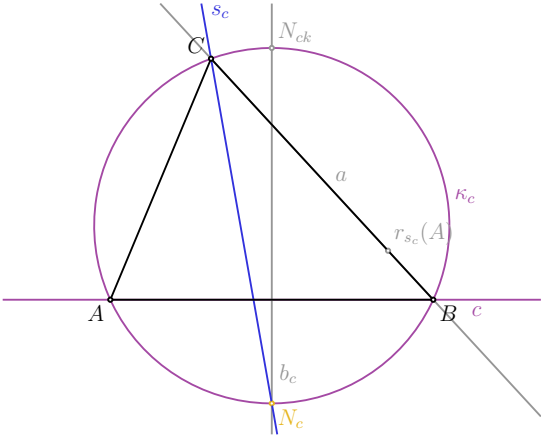


Solution: a, b, b_a

Challenges: Additional objects must be constructed

Example 4

Construct a triangle ABC given its side c, an internal angle bisector s_c, and its circumcircle κ_c

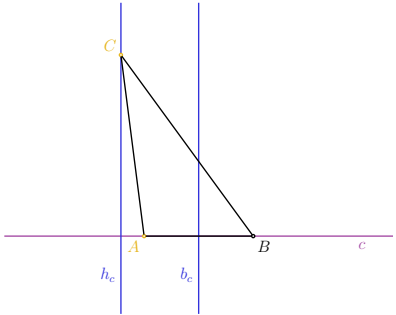


Solution: c, s_c, κ_c

Challenges: Mutual dependency of the objects

Example 5

Construct a triangle ABC given its side c, a side bisector b_c and an altitude h_c

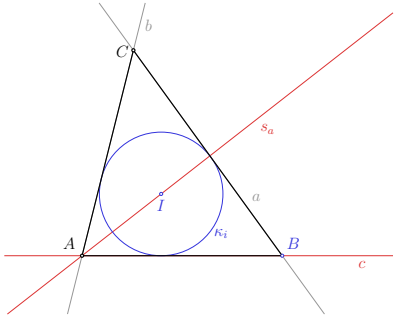


Solution: c, b_c, h_c

Challenges: A construction can start in different ways

Example 6

Construct a triangle ABC given its side c, the internal angle bisector s_a and its incircle κ_i



Solution: c, s_a, κ_i

A construction can start in different ways:

1. one can choose freely lines c and s_a and then a circle κ_i tangent to c with center on s_a;
2. one can choose freely line c, then choose circle κ_i tangent to c, and afterwards a line s_a that passes through the center of circle κ_i, etc.

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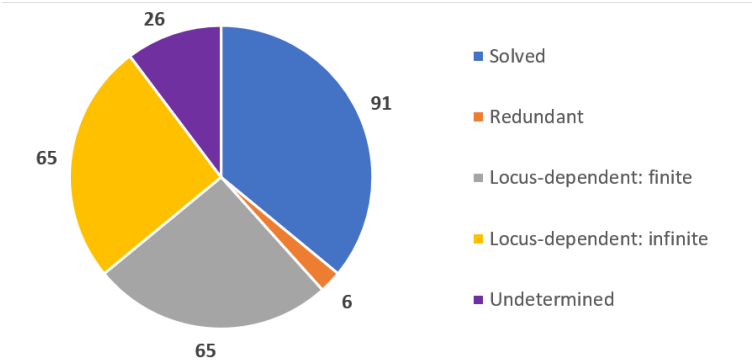
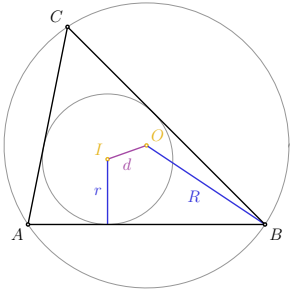
Performance of ArgoTriCS on DTCP corpus

- Total number of problems: $\binom{17}{3} = 680$, with 253 significantly different
- The proportion of LD problems is considerably higher compared to Wernick's corpus



Conclusions and future work

- Resolve the status of remaining problems
- Handle LD problems requiring arbitrary selection of non-vertex elements (point, line, circle)
- Address the problem of constructing a triangle when its incircle and circumcirle are given



Euler's theorem: $d^2 = R(R - 2r)$

- Extend DTCP corpus by additional significant lines and circles

- Thank you for your attention!