Proving theorems on regular polygons by elimination — the elementary way

Zoltán Kovács

PHDL, Linz, Austria + RISC, Hagenberg, Austria

ADG Conference, 2 August 2025
CADE Stuttgart, Germany

Abstract

We use elementary geometry and elimination to prove some non-trivial theorems on regular polygons, including regular 5-, 7- and 9-gons. The required knowledge to follow the proofs does not go beyond high school knowledge.

Background

"Intelligent algorithms" play a growing role in mathematics education. *Asking AI* is already a part of our everyday life. The obtained answers are, however, not always satisfactory; at least, they cannot really be reproduced by human verifiable steps.
ADG brings algorithms to mathematics education that are verifiable. Some of them, of course, are difficult to verify: they may remain black-box algorithms. Elimination (from algebraic geometry) may be such an "instrument". Even though, it can be an extremely useful instrument to bring automated reasoning to young learners quite close.
In this talk we use *elimination as an instrument* in geometric proofs, in particular in those theorems that deal with regular polygons. We only use the Pythagorean theorem as background knowledge.

A possible definition of a regular n-gon

…by using algebraic geometric means

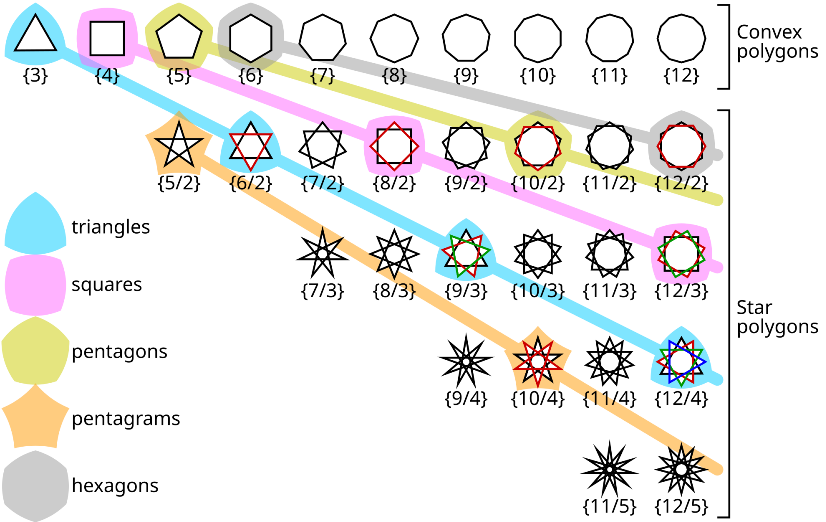
Consider the plane $\mathbb{R}^2$. Put the first two vertices of a regular n-gon in $(0,0)$ and $(1,0)$, and then set up a minimal polynomial $C_n(x) \in \mathbb{Z}[x]$ of $\cos\frac{2\pi}{n}$. Now, by setting $x^2+y^2=1$ and for the coordinates (x_i, y_i) of the regular n-gon,

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} - \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \cdot \left(\begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} - \begin{pmatrix} x_{i-2} \\ y_{i-2} \end{pmatrix} \right),$$

$i = 2, 3, \ldots, n-1$, we have a set of algebraic equations that uniquely describe a regular n-gon together with its star-regular counterparts, as a total of $\varphi(n)$ cases.

A general issue (unsolvable in algebraic geometry)

Impossible to distinguish between regular and star-regular cases



Other issues

- Multiplication of matrices may be a non-trivial concept for young learners.
- The approach also uses a non-trivial formula introduced by Watkins and Zeitlin (1993) which is based on the Chebyshev polynomials of the first kind.

Thus, it makes difficult to communicate even simple results based on regular polygons for a non-expert audience (e.g., for students or young learners).

Properties of regular polygons

. . .that may be used to find other characterizations

- D_1 Their sides are of equal length.
- D_2 Their shortest diagonals are of equal length.
- D_3 Their second-shortest diagonals are of equal length.
- $D_{\dots}$ . . .
- $D_{\lfloor \frac{n}{2} \rfloor}$ Their longest diagonals are of equal length.
- N_i The ith point is different from the first point ($P_i \neq P_0$, $0 \leq i < n$).
- N_{ij} The ith and jth points are different ($P_i \neq P_j$, $0 \leq i, j < n$).

Some possible characterizations

- For $n = 3$: $N_1 \wedge D_1$ $(P_0 \neq P_1 \wedge |P_0P_1| = |P_1P_2| = |P_2P_0|)$
- For $n = 4$: $N_1 \wedge N_2 \wedge D_1 \wedge D_2$ $(P_0 \neq P_1 \wedge P_0 \neq P_2 \wedge |P_0P_1| = |P_1P_2| = |P_2P_3| = |P_3P_0| \wedge |P_0P_2| = |P_1P_3|)$
- For $n = 5$: $N_1 \wedge D_1 \wedge D_2$
- For $n = 6$: $N_1 \wedge N_2 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_3$
- For $n = 7$: $N_1 \wedge D_1 \wedge D_2$

An insufficient (ambiguous) definition

Using GeoGebra as a well-known computer algebra system

CAS

1	x0:=0 → $\mathbf{x0} := \mathbf{0}$
2	y0:=0 → $\mathbf{y0} := \mathbf{0}$
3	x1:=1 → $\mathbf{x1} := \mathbf{1}$
4	y1:=0 → $\mathbf{y1} := \mathbf{0}$
5	p0:=(x1-x0)^2+(y1-y0)^2=(x2-x1)^2+(y2-y1)^2 → $\mathbf{p0}:\ 1 = y2^2 + (x2-1)^2$
6	p1:=(x2-x1)^2+(y2-y1)^2=(x3-x2)^2+(y3-y2)^2 → $\mathbf{p1}:\ y2^2 + (x2-1)^2 = (-x2+x3)^2 + (-y2+y3)^2$
7	p2:=(x3-x2)^2+(y3-y2)^2=(x0-x3)^2+(y0-y3)^2 → $\mathbf{p2}:\ (-x2+x3)^2 + (-y2+y3)^2 = (-x3)^2 + (-y3)^2$
8	p3:=(x2-x0)^2+(y2-y0)^2=(x3-x1)^2+(y3-y1)^2 → $\mathbf{p3}:\ x2^2 + y2^2 = y3^2 + (x3-1)^2$
9	Eliminate({p0,p1,p2,p3},{x3,y3}) → $\{y2\,x2^2 - y2\,x2, x2^3 - x2^2, y2^2 + x2^2 - 2\,x2\}$
10	Solve($9,{x2, y2}) → $\{\{x2 = 0, y2 = 0\}, \{x2 = 1, y2 = 1\}, \{x2 = 1, y2 = -1\}\}$

For $n = 4$, $N_1 \wedge D_1 \wedge D_2$ is insufficient.

Adding N_2 makes the definition unambiguous.

A possible definition of “regular polygons” in school

...by using basic algebraic geometry (only the Pythagorean theorem)

Let $n \geq 3$, the points A_0, A_1, ..., A_{n-1} (elements of $\mathbb{R}^2$) and the assumptions

- N_k for each $1 \leq k < n$,
- D_1,
- D_2,
- ...,
- $D_{\lfloor \frac{n}{2} \rfloor}$

given. Then, we say, $A_0 A_1 \ldots A_{n-1}$ form a regular (star) n-gon.

A possible definition of "regular polygons" in school

...by using basic algebraic geometry (only the Pythagorean theorem)

Let $n \geq 3$, the points A_0, A_1, ..., A_{n-1} (elements of $\mathbb{R}^2$) and the assumptions

- N_k for each $1 \leq k < n$,
- D_1,
- D_2,
- ...,
- $D_{\lfloor \frac{n}{2} \rfloor}$

given. Then, we say, $A_0 A_1 \ldots A_{n-1}$ form a regular (star) n-gon.

Sometimes, we may leave certain assumptions, since some of them are usually unnecessary.

Ratio of a diagonal and a side in a regular pentagon

We drop N_2, N_3, N_4: they are unnecessary

```
0>> n:=5
1>> X:=[0,1,x2,x3,x4]
2>> Y:=[0,0,y2,y3,y4] // Assume N1
3>> p:=[undef]
4>> sqdist(i,j):=(X[j]-X[i])^2+(Y[j]-Y[i])^2
5>> l:=0
6>> for(j=1;j<n;j++) {for(k=1;k<=n/2;k++,l++)
↪  {p[l]:=sqdist(j,irem(j+k,n))-sqdist(0,k)}}
[(x2-1)^2+y2^2-1,(x3-1)^2+y3^2-x2^2-y2^2,(x3-x2)^2+(y3-y2)^2-1,
↪  (x4-x2)^2+(y4-y2)^2-x2^2-y2^2,(x4-x3)^2+(y4-y3)^2-1,
↪  (-x3)^2+(-y3)^2-x2^2-y2^2,(-x4)^2+(-y4)^2-1,
↪  (1-x4)^2+(-y4)^2-x2^2-y2^2]
7>> el1:=eliminate(concat(p,d^2-x2^2-y2^2),[x2,y2,x3,y3,x4,y4])
[-d^4+3*d^2-1]
8>> solve(el1,d)
list[1/2*(-sqrt(5)-1),1/2*(-sqrt(5)+1),1/2*(sqrt(5)-1),1/2*(sqrt(5)+1)]
```

The same code in GeoGebra is longer (but it is possible to implement the same content).

A regular heptagon: exact coordinates

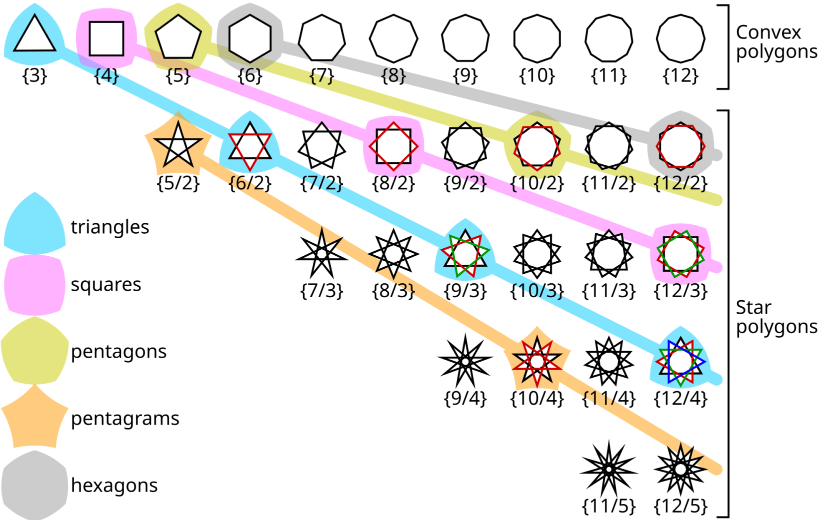
We drop N_2, N_3, N_4, N_5, N_6: they are unnecessary

```
0>> n:=7
1>> X:=[0,1,x2,x3,x4,x5,x6]
2>> Y:=[0,0,y2,y3,y4,y5,y6]
3>> p:=[undef]
4>> sqdist(i,j):=(X[j]-X[i])^2+(Y[j]-Y[i])^2
5>> l:=0
6>> for(j=1;j<n;j++) {for(k=1;k<=n/2;k++,l++) {p[l]:=sqdist(j,irem(j+k,n))-sqdist(0,k)}}
[(x2-1)^2+y2^2-1,(x3-1)^2+y3^2-x2^2-y2^2,(x4-1)^2+y4^2-x3^2-y3^2,
↪  (x3-x2)^2+(y3-y2)^2-1,(x4-x2)^2+(y4-y2)^2-x2^2-y2^2,
↪  (x5-x2)^2+(y5-y2)^2-x3^2-y3^2,(x4-x3)^2+(y4-y3)^2-1,
↪  (x5-x3)^2+(y5-y3)^2-x2^2-y2^2,(x6-x3)^2+(y6-y3)^2-x3^2-y3^2,
↪  (x5-x4)^2+(y5-y4)^2-1,(x6-x4)^2+(y6-y4)^2-x2^2-y2^2,
↪  (-x4)^2+(-y4)^2-x3^2-y3^2,(x6-x5)^2+(y6-y5)^2-1,(-x5)^2+(-y5)^2-x2^2-y2^2,
↪  (1-x5)^2+(-y5)^2-x3^2-y3^2,(-x6)^2+(-y6)^2-1,(1-x6)^2+(-y6)^2-x2^2-y2^2,
↪  (x2-x6)^2+(y2-y6)^2-x3^2-y3^2]
7>> el1:=eliminate(p,[y2,x3,y3,x4,y4,x5,y5,x6,y6])
[-x2^3+5/2*x2^2-3/2*x2+1/8]
8>> fsolve(el1,[x2])
[[0.777479066044],[1.62348980186],[0.0990311320976]]
```

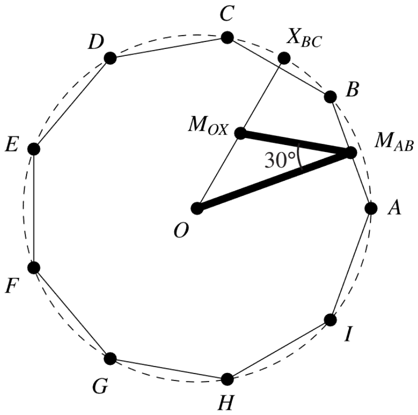
Note: A regular heptagon cannot be constructed via straightedge and compass (that is, with Euclidean means). The obtained polynomial in x_2 is an irreducible cubic.

Recalling {7}, {7/2} and {7/3}

The reason behind there are multiple solutions



A regular nonagon: Karst's theorem



Given a regular nonagon, let M_{AB} be the midpoint of one side, X_{BC} be the mid-arc point of the arc connecting an adjacent side, and M_{OX} the midpoint of OX_{BC}. Then, amazingly, $\angle OM_{AB}M_{OX} = 30°$.

A regular nonagon: Karst's theorem

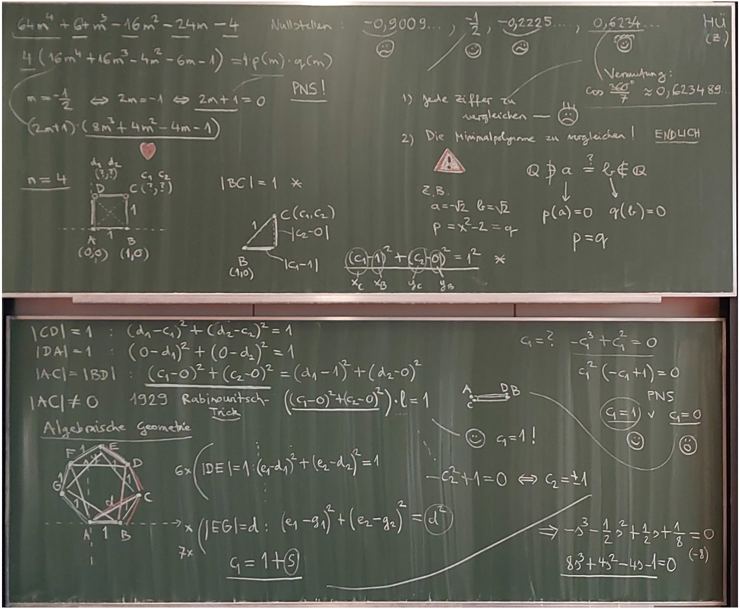
We use $N_1 \wedge N_3 \wedge D_1 \wedge D_2 \wedge D_3 \wedge D_4$ (D_4 may be removed)

```
0>> n:=9
1>> X:=[0,1,x2,x3,x4,x5,x6,x7,x8,ox,mabx,xbcx,moxx]
2>> Y:=[0,0,y2,y3,y4,y5,y6,y7,y8,oy,maby,xbcy,moxy] // N1
3>> p:=[undef]
4>> sqdist(i,j):=(X[j]-X[i])^2+(Y[j]-Y[i])^2
5>> sprod(i,j,k):=(X[j]-X[i])*(X[j]-X[k])+(Y[j]-Y[i])*(Y[j]-Y[k])
6>> midpx(i,j,k):=X[i]-(X[j]+X[k])/2
7>> midpy(i,j,k):=Y[i]-(Y[j]+Y[k])/2
8>> l:=0
9>> for(j=1;j<n;j++) {for(k=1; k<n/2; k++, l++) {p[l]:=sqdist(j,irem(j+k,n))-sqdist(0,k) }}
10>> p[l]:=sqdist(0,3)*t-1 // N3
11>> q:=[undef]
12>> q[0]:=sqdist(0,9)-sqdist(1,9)
13>> q[1]:=sqdist(1,9)-sqdist(2,9)
14>> q[2]:=midpx(10,0,1)
15>> q[3]:=midpy(10,0,1)
16>> q[4]:=sqdist(11,1)-sqdist(11,2)
17>> q[5]:=sqdist(11,9)-sqdist(0,9)
18>> q[6]:=midpx(12,0,11)
19>> q[7]:=midpy(12,0,11)
20>> q[8]:=(sprod(9,10,12))^2-cc*sqdist(9,10)*sqdist(10,12)
21>> el1:=eliminate(p,[y2,x3,y3,x4,y4,x5,y5,x6,y6,x7,y7,x8,y8,t])
[-x2^3+3*x2^2-9/4*x2+1/8]
22>> el2:=eliminate(concat(p,q),[x2,y2,x3,y3,x4,y4,x5,y5,x6,y6,x7,y7,
↪  x8,y8,xbcx,xbcy,ox,oy,mabx,maby,moxx,moxy,t])
[-cc^2+cc-3/16]
23>> fa2:=factor(el2)
[-(4*cc-3)*(4*cc-1)/16]
24>> csolve(fa2,cc)
list[1/4,3/4]
```

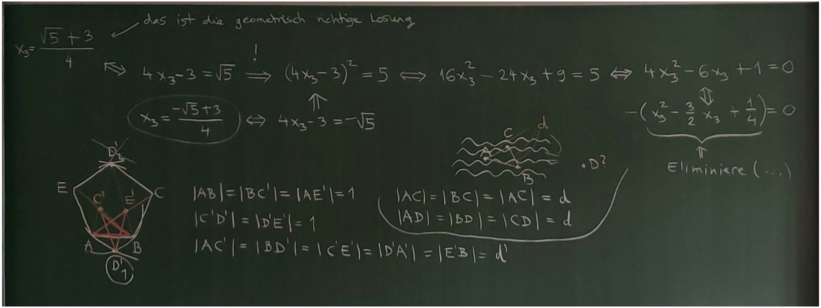
Report on classroom uses

The suggested approach was partially tried in a teacher training course at the author's institute in the winter semester of 2024/2025 among two groups of students (altogether 30 prospective mathematics teachers), with a good success. After the lecturer's explanation, many of the students managed to find the appropriate equations with GeoGebra and they found the coordinates of all searched vertices in a regular pentagon, most of them with no help or after just a few further advices.

Report on classroom uses



Report on classroom uses



Acknowledgments

Many thanks to Bernard Parisse for his continuous help in fine-tuning Giac to get the required results the best way. Also, the author thanks Tomás Recio for his advices regarding to an early version of this contribution.

Grateful thanks to the Research Institute of Symbolic Computation (RISC) in Hagenberg, Austria, for sponsoring the author in participation at ADG 2025.

THANK YOU!

References I

G. Gévay, B. Kovács, and Z. Kovács.
Is a regular polygon determined by its diagonals?, May 2025.
Extended abstract, submission to ADG 2025.

F. Ghourabi, T. Ida, and K. Takahashi.
Logical and algebraic views of a knot fold of a regular heptagon.
In *Proceedings of the International Symposium on Symbolic Computation in Software Science (SCSS2013), Ser. EPiC Series*, volume 15, 06 2013.

Z. Kovács.
Automated detection of interesting properties in regular polygons.
Mathematics in Computer Science, 14:727–755, 2020.

Z. Kovács.
Teaching algebraic curves for gifted learners at age 11 by using LEGO linkages and GeoGebra.
International Journal of Technology in Mathematics Education, 27(4), 2020.

Z. Kovács, T. Recio, and M. P. Vélez.
Reasoning about linkages with dynamic geometry.
Journal of Symbolic Computation, 97:16–30, 2020.

J. Rabinowitsch.
Zum Hilbertschen Nullstellensatz.
Mathematische Annalen, 102(1):520, 1929.

References II

P. Richard, F. Venant, and M. Gagnon.
Issues and challenges in instrumental proof.
In G. Hanna, D. Reid, and M. de Villiers, editors, *Proof Technology in Mathematics Research and Teaching. Mathematics Education in the Digital Era, vol. 14*. Springer, 2019.

B. Scimemi.
Draw of a regular heptagon by the folding.
In H. Huzita, editor, *Proceedings of the First International Meeting of Origami Science and Technology*, pages 71–77, Ferrara, Italy, 1989.

W. Watkins and J. Zeitlin.
The minimal polynomial of $\cos(2\pi/n)$.
The American Mathematical Monthly, 100(5):471–474, 1993.

E. W. Weisstein.
"Regular nonagon." From MathWorld–a Wolfram web resource, 2025.
https://mathworld.wolfram.com/RegularNonagon.html.